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Dichotomies in Equilibrium Computation, and Membership of PLC markets in FIXP*

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Abstract: Piecewise-linear, concave (PLC) utility functions play an important role in work done at the intersection of economics and algorithms. We prove that the problem of computing an equilibrium in Arrow-Debreu markets with PLC utilities and PLC production sets is in the class FIXP. Recently it was shown that these problems are also FIXP-hard [21], hence settling the complexity of this long-standing open problem. Central to our proof is capturing equilibria of these markets as fixed points of a continuous function via a nonlinear complementarity problem (NCP) formulation.

Next, we provide dichotomies for equilibrium computation problems, both Nash and market. There is a striking resemblance in the dichotomies for these two problems, hence providing a unifying view. We note that in the past, dichotomies have played a key role in bringing clarity to the complexity of decision and counting problems.

1 Introduction

Piecewise-linear, concave (PLC) utility functions play a central role in work done at the intersection of economics and algorithms for the following reasons. In economics, it is customary to assume that utility functions are concave, and production sets are convex – this leads to several nice properties such as decreasing marginal utilities and convexity in optimal bundles; the latter is indispensable for applying fixed point theorems for proving existence of equilibria (and for obtaining algorithms for computing them). Since in computer science we assume a finite precision model of computation, differentiable utility functions, which assume infinite precision arithmetic, are not appropriate, thereby making PLC utility functions important.

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Additionally, we will assume that production sets are polyhedral. We call this PLC production since the boundary of polyhedral production set can be defined by a PLC correspondence. Clearly by making the pieces fine enough, the approximation to the original utilities and production sets can be made as good as needed.

Over the last decade and a half, considerable insight has been gained on the complexity of computing equilibria in markets with utility functions being subclasses of PLC functions [14, 4, 37, 7, 22, 19]. However, the question of pinning down the complexity of PLC markets has remained open. In this paper, we take a step towards resolving it by showing that the problem of computing an equilibrium in Arrow-Debreu markets with PLC utilities and PLC production sets is in the class FIXP of Etessami and Yannakakis [16]; recently it was show that these problems are also FIXP-hard [21], hence settling the complexity of this long-standing open problem. Etessami and Yannakakis have also shown that the class FIXP captures the complexity of computing an equilibrium for k-player Nash, henceforth denoted k-Nash, for $k \geq 3$ [16].

Central to our proof is capturing equilibria of these markets as fixed points of a continuous function via a nonlinear complementarity problem (NCP) formulation. We note that at present very few problems have been shown to be in FIXP and we believe this technique, using an NCP formulation, will find use in the future.

Impressive results obtained over the last decade and a half on equilibrium computation have resulted in deep insights; however, these results may seem disparate and there is a need to find a unifying picture to gain even better understanding. In this paper, we attempt this by organizing some of the most important results into dichotomies. We start by observing that the results already known on Nash equilibrium lead to a dichotomy that respects three different criteria, computational complexity being one of them, see Table 1. In a nutshell, this dichotomy establishes a qualitative difference between 2-Nash and k-Nash for $k \geq 3$. The two results stated above, together with other results, lead to analogous dichotomies for market equilibrium.

In the endeavor within TCS, to classify natural computational problems by their complexity, dichotomies have played a key role in bringing much clarity; these dichotomies characterize how the complexity of a certain problem changes as a certain parameter is changed. Perhaps the most well known of these dichotomies is Schaefer's theorem, which gives a complete characterization of when a restriction of SAT, defined via relations over the Boolean domain, is in P and when it is NP-complete. Following this result, a lot of work was done on dichotomies for decision problems, e.g., see [2, 10], and for counting problems, see the extensive survey [3]; in the latter case the dichotomy is between P and #P-complete.

An early result of Megiddo and Papadimitriou [28] indicated that the computability of equilibrium problems should have its own unique character, different from, say, the computability of decision or counting problems. They showed that Nash Equilibrium cannot be NP-hard unless NP = co-NP, and hence new classes were needed to establish evidence of intractability for this problem. The classes PPAD [31] and FIXP [16] have been invaluable in this respect. The dichotomies identified in this paper, which are quite different from those of decision and counting problems, also help highlight the unique character of equilibrium computation.

1.1 The classes PPAD and FIXP

The two complexity classes PPAD, defined by Papadimitriou [31], and FIXP, defined by Etessami and Yannakakis [16], have played an important role in the theory of equilibrium computation. e.g., they capture the complexity of 2-Nash and k-Nash, for $k \geq 3$, respectively. These classes appear to be quite disparate – whereas solutions to problems in the former are rational numbers, those to the latter are algebraic numbers. And whereas the former is contained in function classes NP $\cap$ co-NP, the latter lies somewhere between P and PSPACE, and is likely to be closer to the harder end of PSPACE [38].

Informally, PPAD is the class of problems that allow for "path-following algorithms" and for this reason (see Section 1.2 for a brief discussion on such algorithms), PPAD has an intimate connection with complementary pivot algorithms: obtaining such an algorithm for a problem gives, together with Todd's result [36], membership of the problem in PPAD. Furthermore, the Lemke-Howson result [26], which gave a complementary pivot algorithms for 2-Nash, provided a key motivation for the definition of this class. On the other hand, a problem is in FIXP if its solutions are in one-to-one correspondence with the fixed points of a function which is defined using an arithmetic circuit with operations of $+$, $*$, $/$, max, and an arbitrary number of rational constants.[1]

Another important distinction between the two classes is the nature of approximation allowed: whether the algorithm finds a point x that is almost a fixed point in the sense that $|x - f(x)|$ is small, or the point x is itself close to a fixed point, x^*, i.e. $|x - x^*|$ is small, where by "small" we mean inverse-exponential. Assuming f is Lipschitz continuous, as is the case for all problems considered in this paper, a point satisfying the latter condition also satisfies the former condition; however, a point x satisfying the former condition may be exponentially far away from any fixed point, so the two notions are quite different. Although exact 3-Nash is not in PPAD, if the first kind of approximation is allowed, the problem turns out to be in PPAD; this is a classic result of Daskalakis et. al. [12]. In this case, the strategies computed are within inverse-exponential of being best responses. Yet, this solution may be exponentially far from any exact Nash equilibrium. However, if the second kind of approximation is required, the problem is FIXP-complete and hence unlikely to be in PPAD.

The only results showing membership in FIXP or proving FIXP-hardness for market equilibrium questions we are aware of are the following: [16] prove that the problem of computing an equilibrium in an Arrow-Debreu market is FIXP-complete provided the excess demand is an algebraic function of the prices and this model is a simplified version of the standard model in that individual utility functions are not given, only the aggregate excess demand function is given. [6] show that an Arrow-Debreu market under CES utility functions is in FIXP provided the elasticity parameter for each agent is a rational number $\rho_i < 1$ and is given in unary. [38] show that an Arrow-Debreu market under Leontief utility functions is in FIXP; observe that in the latter cases as well, excess demand is an algebraic function of the prices. No markets with production have been shown to be in FIXP. Further, containment in FIXP for exchange markets with general PLC utilities is also not known. On the other hand exchange markets with even Leontief utilities functions, a simple subclass of PLC utilities, and markets with linear utilities and Leontief production both are shown to be FIXP-hard [21].

For markets under PLC utility functions, considered in this paper, optimal bundles of buyers are not unique. Therefore, excess demand will not be a function, it will be a correspondence – this is one of the major difficulties we need to overcome. For markets with production, the amount of each good available is not a constant, which leads to another difficulty to be overcome.

As stated above, PLC markets may not have rational equilibria and so don't admit a linear complementarity problem (LCP) formulation. Instead, we give a *nonlinear complementarity problem (NCP)* formulation whose solutions are in one-to-one correspondence with market equilibria. We then design a continuous function F over a convex, compact domain which is computable by a FIXP circuit, and we show that the fixed points of F are in one-to-one correspondence with the solutions of the NCP, and hence market equilibria. We believe this technique for proving membership in FIXP using an NCP formulation will find use in the future.

[1]Since all the fixed-points of such a function may be irrational, to remain faithful to Turing machine with discrete-time, class FIXP_a is defined to capture strong approximation where the problem is to find a rational point near to an actual fixed point. Formally, given a function F_C defined by an arithmetic circuit C and an integer k in unary, find a point p such that $||p - p^*||_\infty < 2^{-k}$ for some $p^* = F_C(p^*)$

1.2 Complementary pivot algorithms

An algorithm that walks on the one-skeleton of a polyhedron to find a solution, which is necessarily at a vertex of the polyhedron, is called a *pivoting-based algorithm* or a *path-following algorithm*. The classic example of such an algorithm is the simplex algorithm of Dantzig [11] for linear programming. This algorithm is known to take exponential time in the worst case [24], yet it is the most used algorithm for solving linear programs in practice; moreover, this is despite the fact that linear programming is in P. It turns out that the polynomial time algorithms are too slow in practice.

A pivoting-based algorithm which additionally attempts to satisfy certain complementarity conditions is called a *complementary pivot algorithm*, classic examples being the Lemke-Howson algorithm [26] for 2-Nash and Eaves' algorithm [15], which is based on Lemke's algorithm [25], for the linear case of the Arrow-Debreu market model.

The common feature of these three algorithms is that, similar to the simplex algorithm, they run fast on randomly chosen examples (established in [35, 18], respectively) even though they take exponential time in the worst case (established in [24] and [32] for the first two algorithms and left as an open problem in [18] for the third); the worst case examples are artificially contrived to make the algorithm perform poorly. These algorithms also tend to yield deep structural properties of the underlying problem, e.g., strong duality; index, degree and stability for 2-Nash equilibria [34]; and oddness of number of equilibria [18], respectively. Additionally, complementary pivot algorithms are path-following algorithms and therefore they yield membership of the problem in PPAD.

A FIXP-complete problem is unlikely to have a complementary pivot algorithm since such problems are unlikely to be in PPAD. On the other hand, a PPAD-complete problem may still allow for a fast computability in practice via a complementary pivot algorithm, Hence, the classes PPAD and FIXP are quite different from the viewpoint of computability and it is useful to study the dichotomy between these two classes of problems. We do this in the next section.

2 The Arrow-Debreu Market Model

The Arrow-Debreu market model [1] consists of a set $\mathcal{G}$ of divisible goods, a set $\mathcal{A}$ of agents and a set $\mathcal{F}$ of firms. Let n denote the number of goods in the market.

The production capabilities of a firm is defined by a set of production schedules. If a firm can produce a bundle x^p of goods using bundle x^r as raw material, then such a production schedule defines a production possibility vector (PPV) $(x^p - x^r)$. The set of PPVs of a firm determines its production capabilities. Let $\mathcal{S}^f \in \mathbb{R}^n$ denote the PPV set of firm f. Following are the standard and natural assumptions on $\mathcal{S}^f$ (see [1]).

1. The set $\mathcal{S}^f$ is closed and convex, and contains the origin.
2. The set of produced goods and raw goods of a firm are disjoint. Define $\mathcal{R}^f \stackrel{\text{def}}{=} \{j \in \mathcal{G} \mid v_j < 0,\ v \in \mathcal{S}^f\}$ to be the set of raw goods and $\mathcal{P}^f \stackrel{\text{def}}{=} \{j \in \mathcal{G} \mid v_j > 0,\ v \in \mathcal{S}^f\}$ to be the set of produced goods, then $\mathcal{R}^f \cap \mathcal{P}^f = \emptyset$.[2]
3. *Downward close* - Adding to raw material does not decrease the production, i.e., if $v \in \mathcal{S}^f$ and $w \le v$ then $w \in \mathcal{S}^f$.

[2]Unlike [1] we impose this assumption for simplicity, however as noted in Remark 4.19 the results can be extended to the general case.

4. *No production out of nothing* - Let $\mathcal{S} = \{\oplus_{f\in\mathcal{F}} \mathcal{S}^f\}$ then $\mathcal{S} \cap \mathbb{R}^n_+ = 0$, where for $X,Y \in \mathbb{R}^n$, $X \oplus Y = \{x+y \mid x \in X, y \in Y\}$.

5. *No vacuous production* - Condition $\mathcal{S} \cap -\mathcal{S} = 0$ of [1] implies that no subset of firms can do non-trivial production such that their sum total is 0. Formally, if for $x^f \in \mathcal{S}^f, \forall f$ we have $\sum_f x^f = 0$ then $x^f = 0$, $\forall f$.

The goal of a firm is to produce as per a profit maximizing (optimal) schedule. At prices $p \in \mathbb{R}^n_+$, profit maximizing schedules of firm f are $\arg\max_{x\in\mathcal{S}^f} x \cdot p$. Firms are owned by agents: Θ^i_f is the profit share of agent i in firm f such that $\forall f \in \mathcal{F}$, $\sum_{i\in\mathcal{A}} \Theta^i_f = 1$.

Each agent i comes with an initial endowment of goods; W^i_j is the amount of good j with agent i. The preference of an agent i over bundles of goods is captured by a non-negative, non-decreasing and concave utility function $U_i : \mathbb{R}^n_+ \to \mathbb{R}_+$. Non-decreasingness is due to free disposal property, and concavity captures the law of diminishing marginal returns. Each agent wants to buy a (optimal) bundle of goods that maximizes her utility to the extent allowed by her earned money – from initial endowment and profit shares in the firms; at prices p if ϕ^f is the profit of firm f then money earned by agent i is $\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f$. Without loss of generality, we assume that the total initial endowment of every good is 1, i.e, $\sum_{i\in\mathcal{A}} W^i_j = 1, \forall j \in \mathcal{G}$[3].

Given prices per unit of goods, if there is an assignment of optimal production schedule to each firm and optimal affordable bundle to each agent so that there is no deficiency of any good and goods with surplus amount have zero prices, then such prices are called *market clearing* or *market equilibrium* prices. The market equilibrium problem is to find such prices when they exist. In a celebrated result, Arrow and Debreu [1] proved that a market equilibrium always exists under some mild conditions, however the proof is non-constructive and makes heavy use of Kakutani's fixed point theorem.

A well studied restriction of Arrow-Debreu model is *exchange economy*, i.e., markets without production firms.

3 Dichotomies in Equilibrium Computation

We will assume throughout this paper that all numbers given in an instance are rational. Table 1 gives the dichotomy for Nash equilibrium computation. The rationality of 2-Nash was first established as a corollary of the Lemke-Howson algorithm [26], and the first 3-Nash game having only irrational equilibria was given by Nash [30]. The complexity of finding a Nash equilibrium was shown to be PPAD-complete for two-player games [31, 12, 5], while for three or more it turned out to be FIXP-complete [16]; see Section 1.1 for detailed description of both these classes. However checking if a specific type of equilibrium exists, such as equilibrium with payoff at least h, equilibrium where i^{th} strategy is played with non-zero probability, or simply two or more equilibria, turns out to be NP-complete [23, 8] and ETR-complete [20], respectively.

A well-studied subclass of games is symmetric games, in which all the players have the same set of strategies. Nash [30] showed that such a game always has a symmetric equilibrium, i.e., one in which each player plays the same strategy. The problem of finding such a Nash equilibrium in a symmetric k player game is called symmetric k-Nash. Results analogous to the general case give a similar dichotomy between symmetric 2-Nash and symmetric k-Nash for $k \geq 3$, as shown in Table 2.

Recent results have yielded analogous dichotomies for market equilibrium computation as well, and are presented in Tables 3 and 4, for consumption and production, respectively. These results include the complexity results of [4, 37], establishing PPAD-completeness of computing equilibria for Arrow-Debreu

[3] This is like redefining the unit of goods by appropriately scaling utility and production parameters.

	2-Nash	k-Nash, $k \geq 3$
Nature of solution	Rational [26]	Algebraic; irrational example [30]
Complexity	PPAD-complete [31, 12, 5]	FIXP-complete [16]
Practical algorithms	Lemke-Howson [26]	?
Decision	NP-complete [23, 8]	ETR-complete [33, 20]

Table 1:

	Symmetric 2-Nash	Symmetric k-Nash, $k \geq 3$
Nature of solution	Rational [26]	Algebraic; irrational example [30]
Complexity	PPAD-complete [31, 12, 5]	In FIXP [16]; FIXP-hard [20]
Practical algorithms	Lemke-Howson [26]	?
Decision	NP-complete [8]	ETR-complete [20]

Table 2:

markets under Separable PLC (SPLC) utilities, the new complementary pivot algorithms [18] and [22], and a proof of membership of PLC markets in FIXP, which is established in the current paper. Note that in the tables, results of the current paper have been indicated as $\mathcal{CP}$.

The existence of a market equilibrium is established for a class of markets satisfying certain sufficiency conditions [1, 27]; however, in general a market may or may not have an equilibrium. Similar to the case of Nash equilibrium, the decision problem of checking if a given market has an equilibrium is NP-complete and ETR-complete, respectively, for SPLC and PLC markets [37, 4, 22, 21].

For Table 4, which gives dichotomy for production, we also need to specify the class of utility functions of agents. For this, we have used the following convention. For "negative" results, such as PPAD-hardness, NP-hardness, ETR-hardness, or irrational example, we assume the most restricted utilities, i.e., linear. For "positive" results, such as containment in PPAD or NP, or rationality of equilibria, we assume the most general utilities, i.e., SPLC.

	SPLC utilities	PLC utilities
Nature of solution	Rational [13, 37]	Algebraic [13]; irrational example [15]
Complexity	PPAD-complete [4, 37]	In FIXP: $\mathcal{CP}$ (Theorem 4.6); FIXP-hard [21]
Practical algorithms	GMSV [18] (based on Lemke [25])	?
Decision	NP-complete [37, 4]	ETR-complete [21]

Table 3:

4 Membership in FIXP

In this section, we show that the equilibrium computation problem in markets with PLC utility functions and PLC production functions is in FIXP [16].

We first obtain a characterization of market equilibrium in terms of the solutions of a *nonlinear* complementarity problem[4] (NCP) formulation and then design a continuous function F over a convex and compact

[4]see [9, 29] for the definition of nonlinear complementarity problem.

	SPLC production	PLC production
Nature of solution	Rational [22]	Algebraic: $\mathcal{CP}$ (Theorem 4.9) irrational example [22]
Complexity	PPAD-complete [22]	In FIXP: $\mathcal{CP}$ (Theorem 4.17); FIXP-hard [21]
Practical algorithms	GV [22] (based on Lemke [25])	?
Decision	NP-complete [22]	ETR-complete [21]

Table 4:

domain, computable by a FIXP circuit, i.e., algebraic circuit with $\{max, min, +, -, *, /\}$ operators and rational constants. Further we show that assuming the weakest known sufficiency conditions for the existence of market equilibrium given by Arrow and Debreu [1][5], fixed points of F are in one-to-one correspondence with the solutions of NCP, and hence are related to market equilibria.

Etessami and Yannakakis [16] showed membership in FIXP for exchange markets (markets without production) with explicit algebraic demand function, however this approach does not work for markets with PLC utilities. A major difficulty is that the demand of an agent (or firm) is not an explicit algebraic function of given prices; it is not even unique. The same difficulty was experienced by [37] in proving membership of exchange markets with Separable PLC (SPLC) utilities in PPAD, and they resort to the characterization of PPAD (given in [16]) as a class of exact fixed-point computation problems for polynomial time computable piecewise-linear Brouwer functions. No such characterization for FIXP is known. Further we also consider markets with production firms, which has its own difficulties, like handling market clearing conditions becomes non-trivial due to indefinite quantities of goods in the market.

We develop a novel technique for proving membership in FIXP (PPAD) from NCP (LCP), which may be of independent interest. *The primary intuition for this technique is to use complementarity conditions in the FIXP circuit to push the mapping of any point that is not an equilibrium away from the point itself.* To keep things simple, first we show our result for the exchange markets with PLC utilities, and then extend it to also include PLC production.

4.1 Exchange economy

The piecewise-linear concave (PLC) utility function, of agent i, $u_i : {\mathbb{R}_+}^n \rightarrow \mathbb{R}_+$ can be described as

$$u_i(x^i) = \min_k \{\sum_j U^i_{jk} x^i_j + T^i_k\},$$

where U^i_{jk}'s and T^i_k's are given non-negative rational numbers, and $x^i = (x^i_1, x^i_2, \ldots, x^i_n)$ is a bundle of goods [13]. Given prices per unit p, agent i's optimal bundle is a solution to the following linear program (LP):

$$\begin{aligned} &\max u_i \\ &u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k, \ \forall k \\ &\sum_j x^i_j p_j \leq \sum_j W^i_j p_j \\ &x^i_j \geq 0, \ \forall j \end{aligned} \tag{4.1}$$

[5]We note that Maxfield [27] sufficiency conditions based on economy graph are not suitable for PLC markets.

Let γ^i_k and λ_i be the non-negative dual variables of constraints in the above LP. From the optimality conditions, we get the following linear constraints and complementarity conditions. Note that the constraints are linear assuming prices are given. All variables introduced will have a non-negativity constraint; for the sake of brevity, we will not write them explicitly.

$$\begin{array}{rcr} \forall j: \ \sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j & \text{and} & x^i_j(\sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j) = 0 \\ \forall k: \ u_i \le \sum_j U^i_{jk} x^i_j + T^i_k & \text{and} & \gamma^i_k(u_i - \sum_j U^i_{jk} x^i_j - T^i_k) = 0 \\ \sum_j x^i_j p_j \le \sum_j W^i_j p_j & \text{and} & \lambda_i(\sum_j x^i_j p_j - \sum_j W^i_j p_j) = 0 \\ \sum_k \gamma^i_k = 1 & & \end{array} \tag{4.2}$$

From strong duality, (4.1) and (4.2) are equivalent. Further by taking the sums of complementarity conditions, these conditions also give

$$\begin{array}{rcll} u_i & = & (\sum_k \gamma^i_k) u_i & (\text{Using} \quad \sum_k \gamma^i_k = 1) \\ & = & \sum_{jk} U^i_{jk} x^i_j \gamma^i_k + \sum_k T^i_k \gamma^i_k & (\text{Using} \quad \gamma^i_k(u_i - \sum_j U^i_{jk} x^i_j + T^i_k) = 0) \\ & = & \lambda_i \sum_j x^i_j p_j + \sum_k T^i_k \gamma^i_k & (\text{Using} \quad x^i_j(\sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j) = 0) \\ & = & \lambda_i \sum_j W^i_j p_j + \sum_k \gamma^i_k T^i_k & (\text{Using} \quad \lambda_i(\sum_j x^i_j p_j - \sum_j W^i_j p_j) = 0). \end{array} \tag{4.3}$$

Hence u_i is a redundant variable and can be eliminated using the above expression, however for clarity we keep it as a placeholder variable for the above expression. We get the above constraints for each agent i and all together, they capture the optimal bundle and budget constraints of every agent. At market equilibrium, we also need market clearing of each good, which is essentially, $\sum_i x^i_j \le 1, \ \forall j$. By putting these together and now treating prices p as variables, we get the nonlinear complementarity problem (NCP) formulation as shown in Table 5. Since equilibrium prices are scale invariant in Arrow-Debreu market, we have put $\sum_j p_j = 1$ as well.

$$\begin{array}{rcr} \forall (i,j): \ \sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j & \text{and} & x^i_j(\sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j) = 0 \\ \forall (i,k): \ u_i \le \sum_j U^i_{jk} x^i_j + T^i_k & \text{and} & \gamma^i_k(u_i - \sum_j U^i_{jk} x^i_j - T^i_k) = 0 \\ \forall i: \ \sum_j x^i_j p_j \le \sum_j W^i_j p_j & \text{and} & \lambda_i(\sum_j x^i_j p_j - \sum_j W^i_j p_j) = 0 \\ \forall j: \ \sum_i x^i_j \le 1 & \text{and} & p_j(\sum_j x^i_j - 1) = 0 \\ \forall i: \ \sum_k \gamma^i_k = 1 & \text{and} & u_i = \lambda_i \sum_j W^i_j p_j + \sum_k \gamma^i_k T^i_k \\ \sum_j p_j = 1 & & \end{array}$$

Table 5: E-NCP

The next lemma follows from the above analysis.

Lemma 4.1. *If (p,x,λ,γ) is a solution of E-NCP, then (p,x) is a market equilibrium. Further if (p,x) is a market equilibrium, then $\exists(\lambda,\gamma)$ such that (p,x,λ,γ) is a solution of E-NCP.*

Sufficiency Conditions. Market equilibrium may not exist, and it is NP-complete to decide whether there exists an equilibrium even in markets with SPLC utility functions [37]. Arrow-Debreu [1] showed that a market equilibrium exists under the following sufficiency conditions: endowment matrix $W > 0$ and each agent is non-satiated. In case of PLC utilities, non-satiation condition implies that for every k, there exists a j such that $U^i_{jk} > 0$.

Next we define a continuous function $F : D \to D$, where D is convex and compact and show that the fixed points of F are in one-to-one correspondence with the solutions of E-NCP, and hence are related to market equilibria using Lemma 4.1. Since F is continuous on a convex and compact D, there exists a fixed point. Clearly, for such a theorem, we need to assume sufficiency conditions.

To define D, first we obtain upper bounds on all variables at equilibrium. Let $x_{max} \stackrel{\text{def}}{=} 1.1$, $W_{min} \stackrel{\text{def}}{=} \min_{(i,j)} W^i_j$, $U_{max} \stackrel{\text{def}}{=} \max_{(i,j,k)} U^i_{jk}$, $T_{max} \stackrel{\text{def}}{=} \max_{(i,k)} T^i_k$, and $\lambda_{max} \stackrel{\text{def}}{=} 2n^{(U_{max}+T_{max})}/_{W_{min}}$. Note that $W_{min} > 0$ under sufficiency conditions. Since the total quantity of every good is 1, $0 \le x^i_j < x_{max}$ at equilibrium. Using (4.3), we get

$$\lambda_i = \frac{u_i - \sum_k \gamma^i_k T^i_k}{\sum_j W^i_j p_j} \le \frac{u_i}{W_{min}} \le \frac{\sum_j U^i_{jk} x^i_j + T^i_k}{W_{min}} < \frac{2n(U_{max}+T_{max})}{W_{min}} = \lambda_{max} \text{ at equilibrium.}$$

Let $D \stackrel{\text{def}}{=} \{(p,x,\gamma,\lambda) \in \mathbb{R}^N_+ \mid \sum_j p_j = 1;\ x^i_j \le x_{max};\ \sum_k \gamma^i_k = 1;\ \lambda_i \le \lambda_{max}\}$, where N is the total number of variables, and let $(\overline{p},\overline{x},\overline{\gamma},\overline{\lambda}) \stackrel{\text{def}}{=} F(p,x,\gamma,\lambda)$ as given in Table 6.

$$\boxed{\begin{aligned}
\overline{p}_j &= \frac{p_j + \max\{\sum_i x^i_j - 1, 0\}}{\sum_l (p_l + \max\{\sum_i x^i_l - 1, 0\})} \\
\overline{\gamma}^i_k &= \frac{\gamma^i_k + \max\{u_i - \sum_j U^i_{jk} x^i_j - T^i_k, 0\}}{\sum_a (\gamma^i_a + \max\{u_i - \sum_j U^i_{ja} x^i_j - T^i_a, 0\})} \\
\overline{x}^i_j &= \min\left\{\max\left\{x^i_j + \textstyle\sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j, 0\right\}, x_{max}\right\} \\
\overline{\lambda}_i &= \min\left\{\max\left\{\lambda_i + \textstyle\sum_j x^i_j p_j - \sum_j W^i_j p_j, 0\right\}, \lambda_{max}\right\}
\end{aligned}}$$

Table 6: FIXP Circuit for Exchange Economy

The following claim is straightforward using Lemma 4.1 and we omit its proof.

Claim 4.2. *Every market equilibrium gives a fixed point of F.*

Next assuming sufficiency conditions for the existence of market equilibrium, we show that every fixed point of F gives a market equilibrium. Table 7 gives all the conditions that might lead to a fixed point of F based on the update rules in Table 6.

Next we show that none of the conditions in shaded rows, namely (1.2), (2.2), (3.3), and (4.3), are satisfied at fixed points of F, which implies that each fixed point of F gives a solution of E-NCP, and hence market equilibrium.

$\overline{p} = p$	case 1:	$\sum_i x^i_j \leq 1,\ \forall j$	(1.1)
	case 2:	If $p_j = 0$, then $\sum_i x^i_j \leq 1$ If $p_j > 0$, then $\sum_i x^i_j > 1$	(1.2)
$\overline{\gamma}^i = \gamma^i$	case 1:	$u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k,\ \forall k$	(2.1)
	case 2:	If $\gamma^i_k = 0$, then $u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k$ If $\gamma^i_k > 0$, then $u_i > \sum_j U^i_{jk} x^i_j + T^i_k$	(2.2)
$\overline{x}^i_j = x^i_j$	$x^i_j = 0$	$\sum_k U^i_{jk} \gamma^i_k \leq \lambda_i p_j$	(3.1)
	$0 < x^i_j < x_{max}$	$\sum_k U^i_{jk} \gamma^i_k = \lambda_i p_j$	(3.2)
	$x^i_j = x_{max}$	$\sum_k U^i_{jk} \gamma^i_k \geq \lambda_i p_j$	(3.3)
$\overline{\lambda_i} = \lambda_i$	$\lambda_i = 0$	$\sum_j x^i_j p_j \leq \sum_j W^i_j p_j$	(4.1)
	$0 < \lambda_i < \lambda_{max}$	$\sum_j x^i_j p_j = \sum_j W^i_j p_j$	(4.2)
	$\lambda_i = \lambda_{max}$	$\sum_j x^i_j p_j \geq \sum_j W^i_j p_j$	(4.3)

Table 7: Conditions for a Fixed Point Based on the Update Rules in Table 6

Claim 4.3. *At every fixed point of F, $0 < \lambda_i < \lambda_{max}, \forall i$.*

Proof. First suppose that $\lambda_i = \lambda_{max}$ for some i at a fixed point. It implies that for every good j such that $p_j \geq \frac{W_{min}}{2n}$, we have $x^i_j = 0$ (from (3.1)). Hence, $\sum_j x^i_j p_j < W_{min}$, which contradicts $\sum_j x^i_j p_j \geq \sum_j W^i_j p_j$ (from (4.3)). Hence $\lambda_i < \lambda_{max}, \forall i$ at a fixed point.

Next suppose that $\lambda_i = 0$ for some i at a fixed point. It implies that for every $\gamma^i_k > 0$ and $U^i_{jk} > 0$, we have $x^i_j = x_{max}$ (from (3.3)). Note that here we use the *sufficiency condition* that for every k there exists a j such that $U^i_{jk} > 0$. Further $p_j > 0$ for such goods and $\sum_i x^i_j > 1$ for all goods whose $p_j > 0$ (from (1.2)). By this, we get $\sum_{i,j} x^i_j p_j > 1 = \sum_{i,j} W^i_j p_j$. This further implies that $\exists i'$ such that $\sum_j x^{i'}_j p_j > \sum_j W^{i'}_j p_j$ and $\lambda_{i'} = \lambda_{max}$, which is a contradiction. □

Claim 4.4. *At every fixed point of F, $\sum_i x^i_j \leq 1, \forall j$.*

Proof. Suppose $\exists j$ such that $\sum_i x^i_j > 1$. It implies that $p_j > 0$ (from (1.2)). This further implies that whenever $p_j > 0$, we have $\sum_i x^i_j > 1$. Hence, we have $\sum_{ij} x^i_j p_j > 1 = \sum_{i,j} W^i_j p_j$. By this, we get that $\exists i'$ such that $\sum_j x^{i'}_j p_j > W^{i'}_j p_j$ and hence $\lambda_{i'} = \lambda_{max}$ (from (4.3)), contradicting Claim 4.3. □

Note that Claim 4.4 implies that $x^i_j < x_{max}, \forall (i,j)$ at every fixed point of F.

Claim 4.5. *At every fixed point of F, $u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k, \forall (i,k)$.*

Proof. Note that u_i is a placeholder variable for $\lambda_i \sum_j W^i_j p_j + \sum_k \gamma^i_k T^i_k$. Suppose $\exists (i,k)$ such that $u_i > \sum_j U^i_{jk} x^i_j + T^i_k$, then we have $\forall (i,k)$, $\gamma^i_k > 0 \Rightarrow u_i > \sum_j U^i_{jk} x^i_j + T^i_k$ (from (2.2)). This implies that $\sum_k u_i \gamma^i_k > \sum_{j,k} U^i_{jk} x^i_j \gamma^i_k + \sum_k T^i_k \gamma^i_k$. From Claims 4.4 and 4.3, we have $\sum_k U^i_{jk} \gamma^i_k x^i_j = \lambda_i p_j x^i_j, \forall (i,j)$ and $\sum_j x^i_j p_j \lambda_i = \sum_j W^i_j p_j \lambda_i, \forall i$. Putting these together, we get that $u_i > \sum_j W^i_j p_j \lambda_i + \sum_k T^i_k \gamma^i_k$, which is a contradiction. □

Claims 4.3, 4.4, 4.5 imply that none of the conditions (1.2), (2.2), (3.3), (4.3) are satisfied at fixed points of F. Therefore, we get the following theorem.

Theorem 4.6. *Assuming sufficient conditions of the existence of market equilibrium, every fixed point of F gives a solution of E-NCP and hence a market equilibrium. Furthermore, F can be computed by a FIXP-circuit and hence market equilibrium computation problem for PLC utilities is in FIXP.*

Remark 4.7. This technique can be used to obtain a Linear-FIXP (equivalent to PPAD) circuit for markets with SPLC utilities using the linear complementary problem (LCP) formulation given in [18], thereby giving an alternate proof of membership in PPAD for such markets. However, the same approach for proving membership in Linear-FIXP does not seem to work for 2-Nash using its LCP formulation.

4.2 Markets with production

Recall from Section 2 that each firm has a production technology to produce a set of goods from a set of raw goods. The PLC production technology of firm f can be described as

$$\sum_j D^f_{jk} x^{f,p}_j \le \sum_j C^f_{jk} x^{f,r}_j + T^f_k, \ \forall k,$$

where D^f_{jk}'s, C^f_{jk}'s and T^f_k's are given non-negative rational numbers, and $x^{f,p}_j$ and $x^{f,r}_j$ denote the amount of good j produced and used respectively [17]. In the above expression, the first summation is on goods j which can be produced by firm f, and the second summation is on goods j which can be used as a raw material. These two sets of goods are disjoint as described in Section 2, however for simplicity we do not introduce more symbols and taking summation over all goods.

Given prices p, firm f's profit maximizing plan is a solution of the following linear program (LP):

$$\begin{aligned} \max \sum_j p_j x^{f,p}_j - \sum_j p_j x^{f,r}_j& \\ \sum_j D^f_{jk} x^{f,p}_j \le \sum_j C^f_{jk} x^{f,r}_j + T^f_k, \ \forall k& \\ x^{f,p}_j \ge 0, x^{f,r}_j \ge 0& \end{aligned} \tag{4.4}$$

Let δ^f_k be the non-negative dual variables of constraints in the above LP. From the optimality conditions, we get the following linear constraints and complementarity conditions. Note that the constraints are linear assuming prices are given. All variables introduced will have a non-negativity constraint; for the sake of brevity, we will not write them explicitly.

$$\begin{array}{rlr} \forall k: \ \sum_j D^f_{jk} x^{f,p}_j \le \sum_j C^f_{jk} x^{f,r}_j + T^f_k & \text{and} & \delta^f_k (\sum_j D^f_{jk} x^{f,p}_j - \sum_j C^f_{jk} x^{f,r}_j + T^f_k) = 0 \\ \forall j: \ p_j \le \sum_k D^f_{jk} \delta^f_k & \text{and} & x^{f,p}_j (p_j - \sum_k D^f_{jk} \delta^f_k) = 0 \\ \forall j: \ \sum_k C^f_{jk} \delta^f_k \le p_j & \text{and} & x^{f,r}_j (\sum_k C^f_{jk} \delta^f_k - p_j) = 0 \end{array} \tag{4.5}$$

From strong duality, (4.4) and (4.5) are equivalent. Let ϕ^f captures the profit of firm f, i.e., $\phi^f = \sum_j p_j x^{f,p}_j - \sum_j p_j x^{f,r}_j$. Further by taking the sums of complementarity conditions, these conditions also give

$$\phi^f = \sum_k \delta^f_k T^f_k. \tag{4.6}$$

We get the above constraints for each firm k and all together, they capture the optimal production plan of every firm. Next we need to add constraints capturing optimality of the bundles of agents and market clearing. For this, we only need to modify the market clearing constraints in Table 5 appropriately and we get the nonlinear complementarity problem (NCP) formulation AD-NCP for market equilibrium as shown in Table 8.

$$
\begin{array}{rcl}
\forall (f,k):\ \sum_j D^f_{jk} x^{f,p}_j \le \sum_j C^f_{jk} x^{f,r}_j + T^f_k & \text{and} & \delta^f_k\big(\sum_j D^f_{jk} x^{f,p}_j - \sum_j C^f_{jk} x^{f,r}_j - T^f_k\big) = 0 \\
\forall (f,j):\ p_j \le \sum_k D^f_{jk}\delta^f_k & \text{and} & x^{f,p}_j\big(p_j - \sum_k D^f_{jk}\delta^f_k\big) = 0 \\
\forall (f,j):\ \sum_k C^f_{jk}\delta^f_k \le p_j & \text{and} & x^{f,r}_j\big(\sum_k C^f_{jk}\delta^f_k - p_j\big) = 0 \\
\forall (i,j):\ \sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j & \text{and} & x^i_j\big(\sum_l U^i_{jk}\gamma^i_k - \lambda_i p_j\big) = 0 \\
\forall (i,k):\ u_i \le \sum_j U^i_{jk} x^i_j + T^i_k & \text{and} & \gamma^i_k\big(u_i - \sum_j U^i_{jk} x^i_j - T^i_k\big) = 0 \\
\forall i:\ \sum_j x^i_j p_j \le \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f & \text{and} & \lambda_i\big(\sum_j x^i_j p_j - \sum_j W^i_j p_j - \sum_f \Theta^i_f \phi^f\big) = 0 \\
\forall j:\ \sum_i x^i_j + \sum_f x^{f,r}_j \le 1 + \sum_f x^{f,p}_j & \text{and} & p_j\big(\sum_i x^i_j + \sum_f x^{f,r}_j - 1 - \sum_f x^{f,p}_j\big) = 0 \\
\forall i:\ \sum_k \gamma^i_k = 1 & \text{and} & u_i = \lambda_i\big(\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f\big) + \sum_k \gamma^i_k T^i_k \\
\forall f:\ \phi^f = \sum_k \delta^f_k T^f_k & & \\
\sum_j p_j = 1 & &
\end{array}
$$

Table 8: AD-NCP

The next lemma and theorem follow from the construction.

Lemma 4.8. *If $(p,x,x^p,x^r,\lambda,\gamma,\delta)$ is a solution of AD-NCP, then (p,x,x^p,x^r) is a market equilibrium. Furthermore, if (p,x,x^p,x^r) is a market equilibrium, then $\exists(\lambda,\gamma,\delta)$ such that $(p,x,x^p,x^r,\lambda,\gamma,\delta)$ is a solution of AD-NCP.*

Theorem 4.9. *Assuming sufficient conditions of the existence of equilibrium, a market with PLC utilities and PLC production has at least one equilibrium with algebraic prices.*

Sufficiency Conditions. For markets with production, Arrow-Debreu [1] gave the following sufficiency conditions for the existence of equilibrium: endowment matrix $W > 0$, each agent is non-satiated, *no production out of nothing* and *no vacuous production* (defined in Section 2). In case of PLC production, the last two conditions mean that the following linear constraints define a bounded polyhedron.[6]

[6] If the polyhedron is unbounded, then a set of firms together would be able to produce some good in unbounded amount which will give rise to a production cycle along which every good is produced and consumed in unbounded amount. However the *no production out of nothing* and *no vacuous production* conditions imply that along any production cycle some good has to strictly reduce in quantity. Since initial endowment of every good is one, such an unbounded production along a cycle is not possible.

$$\begin{aligned}\forall(f,k):\ & \sum_j D^f_{jk}x^{f,p}_j \le \sum_j C^f_{jk}x^{f,r}_j + T^f_k\\ \forall j:\ & \sum_f x^{f,r}_j \le 1 + \sum_f x^{f,p}_j\\ \forall(f,j):\ & x^{f,p}_j \ge 0;\ \ x^{f,r}_j \ge 0,\end{aligned} \tag{4.7}$$

where first is production constraint and second is supply constraint. Let x^* be the maximum possible value of a variable over these constraints. Note that the bit length of x^* is polynomial in the size of input and can be computed in polynomial time.

Next we define a continuous function $F : D \to D$, where D is convex and compact and show that the fixed points of F are in one-to-one correspondence with the market equilibria. Since F is continuous on a convex and compact D, there exists a fixed point. Clearly, for such a theorem, we need to assume sufficiency conditions.

To define D, first we obtain upper bounds on all variables at equilibrium. Let $x^p_{max} \overset{\text{def}}{=} x^* + 1$ and $x_{max} \overset{\text{def}}{=} 2lx^p_{max} + 2$, where l is the total number of firms and x^* is as discussed above in the sufficiency conditions. Next define *min* and *max* of every input C, D, T, U, W as

$$C_{min} \overset{\text{def}}{=} \min_{(f,j,k)} \{C^f_{jk} \mid C^f_{jk} > 0\} \text{ and } C_{max} \overset{\text{def}}{=} \max_{(f,j,k)} C^f_{jk}.$$

Let $x^r_{max} \overset{\text{def}}{=} x_{max} + (nx^p_{max}{}^{D_{max}}/_{C_{min}})$, $\delta_{max} \overset{\text{def}}{=} \max\{1/C_{min}, 1/D_{min}\} + 1$, and $\lambda_{max} \overset{\text{def}}{=} 4nx_{max}{}^{(U_{max}+T_{max})}/_{W_{min}}$.

Clearly, $x^i_j < x_{max}$, $x^{f,p}_j < x^p_{max}$ and $x^{f,r}_j < x^r_{max}$ at equilibrium. Using $u_i = \lambda_i(\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f) + \sum_k \gamma^i_k T^i_k$, we get $\lambda_i \le nx_{max}{}^{(U_{max}+T_{max})}/_{W_{min}} < \lambda_{max}$ at equilibrium. Using $\sum_k C^f_{jk}\delta^f_k \le p_j$, we get an upper bound on δ^f_k at equilibrium as: $\delta^f_k \le 1/C_{min} < \delta_{max}$. Let

$$D \overset{\text{def}}{=} \{(p,x,x^p,x^r,\gamma,\delta,\lambda) \in \mathbb{R}^N_+ \mid \sum_j p_j = 1;\ x^i_j \le x_{max};\ x^{f,p}_j \le x^p_{max};\ x^{f,r}_j \le x^r_{max};\ \sum_k \gamma^i_k = 1;\ \delta^f_k \le \delta_{max};\ \lambda_i \le \lambda_{max}\},$$

where N is the total number of variables, and let $(\overline{p},\overline{x},\overline{x}^p,\overline{x}^r,\overline{\gamma},\overline{\delta},\overline{\lambda}) \overset{\text{def}}{=} F(p,x,x^p,x^r,\gamma,\delta,\lambda)$ as given in Table 9.

The following claim is straightforward using Lemma 4.8 and we omit its proof.

Claim 4.10. *Every market equilibrium is a fixed point of F.*

Next assuming sufficiency conditions for the existence of market equilibrium, we show that every fixed point of F is a market equilibrium. Table 10 gives all the conditions that might lead to a fixed point of F based on the update rules in Table 9. We show that none of the conditions in shaded rows, namely (1.2), (2.2), (3.3), (4.3), (5.3), (6.3) and (7.3), are satisfied at fixed points of F, which implies that each fixed point of F gives a solution of AD-NCP in Table (8) and hence a market equilibrium.

Claim 4.11. *At every fixed point of F, $\sum_j D^f_{jk}x^{f,p}_j \le \sum_j C^f_{jk}x^{f,r} + T^f_k, \forall(f,k)$.*

Proof. Suppose $\exists(f,k)$ such that $\sum_j D^f_{jk}x^{f,p}_j > \sum_j C^f_{jk}x^{f,r} + T^f_k$, then we have $\delta^f_k = \delta_{max}$ (from (7.3)). This implies that whenever $D^f_{jk} > 0$, we have $x^{f,p}_j = 0$ (from (6.1)), which contradicts the starting assumption. $\square$

$\overline{p}_j = \dfrac{p_j + \max\{\sum_i x^i_j + \sum_f x^{f,r}_j - 1 - \sum_f x^{f,p}_j, 0\}}{\sum_l (p_l + \max\{\sum_i x^i_l + \sum_f x^{f,r}_l - 1 - \sum_f x^{f,p}_l, 0\})}$
$\overline{\gamma}^i_k = \dfrac{\gamma^i_k + \max\{u_i - \sum_j U^i_{jk} x^i_j - T^i_k, 0\}}{\sum_a \gamma^i_a + \max\{u_i - \sum_j U^i_{ja} x^i_j - T^i_a, 0\}}$
$\overline{\delta}^f_k = \min\left\{\max\left\{\delta^f_k + \sum_j D^f_{jk} x^{f,p}_j - \sum_j C^f_{jk} x^{f,r}_j - T^f_k, 0\right\}, \delta_{max}\right\}$
$\overline{x}^i_j = \min\left\{\max\left\{x^i_j + \sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j, 0\right\}, x_{max}\right\}$
$\overline{x}^{f,p}_j = \min\left\{\max\left\{x^{f,p}_j + p_j - \sum_k D^f_{jk}\gamma^i_k, 0\right\}, x^p_{max}\right\}$
$\overline{x}^{f,r}_j = \min\left\{\max\left\{x^{f,r}_j + \sum_k C^f_{jk}\delta^f_k - p_j, 0\right\}, x^r_{max}\right\}$
$\overline{\lambda}_i = \min\left\{\max\left\{\lambda_i + \sum_j x^i_j p_j - \sum_j W^i_j p_j, 0\right\}, \lambda_{max}\right\}$

Table 9: FIXP Circuit for Markets with Production

Claim 4.12. *At every fixed point of F, $\sum_k C^f_{jk}\delta^f_k \le p_j, \forall (f,j)$.*

Proof. Suppose $\exists (f,j)$ such that $\sum_k C^f_{jk}\delta^f_k > p_j$, then we have $x^{f,r}_j = x^r_{max}$ (from (5.3)). It implies that whenever $C^f_{jk} > 0$, we have $\delta^f_k = 0$ (from (7.1)), which contradicts the starting assumption. □

Claim 4.13. *If $\sum_{i,j} x^i_j p_j + \sum_{f,j} x^{f,r}_j p_j > \sum_{i,j} W^i_j p_j + \sum_{f,j} x^{f,p}_j p_j$, then $\exists i$ such that $\sum_j x^i_j p_j > \sum_j W^i_j p_j + \sum_{f,k}\Theta^i_f T^f_k \delta^f_k$.*

Proof. This proof is by contradiction. Suppose we have $\sum_j x^i_j p_j \le \sum_j W^i_j p_j + \sum_{f,k}\Theta^i_f T^f_k \delta^f_k, \forall i$, then summing it over all i and using $\sum_i \Theta^i_f = 1$, we get

$$\sum_{i,j} W^i_j p_j + \sum_{f,k} T^f_k \delta^f_k \ge \sum_{i,j} x^i_j p_j > \sum_{i,j} W^i_j p_j + \sum_{f,j} x^{f,p}_j p_j - \sum_{f,j} x^{f,r}_j p_j \tag{4.8}$$

Claims 4.11 and 4.12 imply that $x^{f,r}_j(\sum_k C^f_{jk}\delta^f_k - p_j) = 0, \forall (f,j)$ and $\delta^f_k(\sum_j D^f_{jk} x^{f,p}_j - \sum_j C^f_{jk} x^{f,r}_j - T^f_k) = 0, \forall (f,k)$, and it further implies that $\sum_{f,k} T^f_k \delta^f_k = \sum_{f,j,k}\delta^f_k D^f_{jk} x^{f,p}_j - \sum_{f,j} x^{f,r}_j p_j$. Using this and (4.8) we get $\sum_{f,j,k}\delta^f_k D^f_{jk} x^{f,p}_j > \sum_{f,j} x^{f,p}_j p_j$, which is a contradiction because (6.1), (6.2) and (6.3) imply that $\sum_{f,j,k}\delta^f_k D^f_{jk} x^{f,p}_j \le \sum_{f,j} x^{f,p}_j p_j$. □

Claim 4.14. *At every fixed point of F,*

- $0 < \lambda_i < \lambda_{max}, \forall i$
- $\sum_i x^i_j + \sum_f x^{f,r}_j \le 1 + \sum_f x^{f,p}_j, \forall j$
- $x^i_j < x_{max}, \forall (i,j)$

$\overline{p} = p$	case 1:	$\sum_i x_j^i + \sum_f x_j^{f,r} \le 1 + \sum_f x_j^{f,p},\ \forall j$	(1.1)
	case 2:	$p_j = 0$ and $\sum_i x_j^i + \sum_f x_j^{f,r} \le 1 + \sum_f x_j^{f,p}$ $p_j > 0$ and $\sum_i x_j^i + \sum_f x_j^{f,r} > 1 + \sum_f x_j^{f,p}$	(1.2)
$\overline{\gamma}^i = \gamma^i$	case 1:	$u_i \le \sum_j U_{jk}^i x_j^i + T_k^i,\ \forall k$	(2.1)
	case 2:	$\gamma_k^i = 0$ and $u_i \le \sum_j U_{jk}^i x_j^i + T_k^i$ $\gamma_k^i > 0$ and $u_i > \sum_j U_{jk}^i x_j^i + T_k^i$	(2.2)
$\overline{x}_j^i = x_j^i$	$x_j^i = 0$	$\sum_k U_{jk}^i \gamma_k^i \le \lambda_i p_j$	(3.1)
	$0 < x_j^i < x_{max}$	$\sum_k U_{jk}^i \gamma_k^i = \lambda_i p_j$	(3.2)
	$x_j^i = x_{max}$	$\sum_k U_{jk}^i \gamma_k^i \ge \lambda_i p_j$	(3.3)
$\overline{\lambda_i} = \lambda_i$	$\lambda_i = 0$	$\sum_j x_j^i p_j \le \sum_j W_j^i p_j + \sum_{f,k} \Theta_f^i \delta_k^f T_k^f$	(4.1)
	$0 < \lambda_i < \lambda_{max}$	$\sum_j x_j^i p_j = \sum_j W_j^i p_j + \sum_{f,k} \Theta_f^i \delta_k^f T_k^f$	(4.2)
	$\lambda_i = \lambda_{max}$	$\sum_j x_j^i p_j \ge \sum_j W_j^i p_j + \sum_{f,k} \Theta_f^i \delta_k^f T_k^f$	(4.3)
$\overline{x}_j^{f,r} = x_j^{f,r}$	$x_j^{f,r} = 0$	$\sum_k C_{jk}^f \delta_k^f \le p_j$	(5.1)
	$0 < x_j^{f,r} < x_{max}^r$	$\sum_k C_{jk}^f \delta_k^f = p_j$	(5.2)
	$x_j^{f,r} = x_{max}^r$	$\sum_k C_{jk}^f \delta_k^f \ge p_j$	(5.3)
$\overline{x}_j^{f,p} = x_j^{f,p}$	$x_j^{f,p} = 0$	$p_j \le \sum_k D_{jk}^f \delta_k^f$	(6.1)
	$0 < x_j^{f,p} < x_{max}^p$	$p_j = \sum_k D_{jk}^f \delta_k^f$	(6.2)
	$x_j^{f,p} = x_{max}^p$	$p_j \ge \sum_k D_{jk}^f \delta_k^f$	(6.3)
$\overline{\delta}_k^f = \delta_k^f$	$\delta_k^f = 0$	$\sum_j D_{jk}^f x_j^{f,p} \le \sum_j C_{jk}^f x_j^{f,r} + T_k^f$	(7.1)
	$0 < \delta_k^f < \delta_{max}$	$\sum_j D_{jk}^f x_j^{f,p} = \sum_j C_{jk}^f x_j^{f,r} + T_k^f$	(7.2)
	$\delta_k^f = \delta_{max}$	$\sum_j D_{jk}^f x_j^{f,p} \ge \sum_j C_{jk}^f x_j^{f,r} + T_k^f$	(7.3)

Table 10: Conditions for a Fixed Point Based on the Update Rules in Table 9

Proof. First suppose that $\lambda_i = \lambda_{max}$ for some i at a fixed point. It implies that for every good j such that $p_j \ge W_{min}/2nx_{max}$, we have $x_j^i = 0$ (from (3.1)). Hence, $\sum_j x_j^i p_j < W_{min}$, which contradicts (4.3). Hence $0 \le \lambda_i < \lambda_{max}, \forall i$ at a fixed point.

Next suppose that $\lambda_i = 0$ for some i at a fixed point. It implies that for every $\gamma_k^i > 0$ and $U_{jk}^i > 0$, we have $x_j^i = x_{max}$ (from (3.3)). Note that here we use the *sufficiency condition* that for every k there exists a j such that $U_{jk}^i > 0$. Since x_{max} is much larger than $1 + \sum_f x_j^{f,p}$, we have $p_j > 0$ for such goods and $\sum_i x_j^i + \sum_f x_j^{f,r} > 1 + \sum_f x_j^{f,p}$ for all goods whose $p_j > 0$ (from (1.2)). By this, we get $\sum_{i,j} x_j^i p_j + \sum_{f,j} x_j^{f,r} p_j >$

$\sum_{i,j} W^i_j p_j + \sum_{f,j} x^{f,p}_j p_j$. Using Claim 4.13, it implies that $\exists i'$ such that $\sum_j x^{i'}_j p_j > \sum_j W^{i'}_j p_j + \sum_{f,k} \Theta^{i'}_f T^f_k \delta^f_k$ and $\lambda_{i'} = \lambda_{max}$ (from (4.3)), which is a contradiction.

Finally suppose that there exists a j such that $\sum_i x^i_j + \sum_f x^{f,r}_j > 1 + \sum_f x^{f,p}_j$, then we have $p_j > 0$ and whenever $p_j > 0$, $\sum_i x^i_j + \sum_f x^{f,r}_j > 1 + \sum_f x^{f,p}_j$ (from (1.2)). It implies that there exists an i such that $\lambda_i = \lambda_{max}$, which is a contradiction. This also implies that $x^i_j < x_{max}, \forall (i,j)$. □

The proof of next claim is similar as in Claim 4.5, hence omitted.

Claim 4.15. *At every fixed point of F, $u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k, \forall (i,k)$.*

Claim 4.16. *At every fixed point of F, $p_j \leq \sum_k D^f_{jk} \delta^f_k, \forall (f,j)$.*

Proof. Suppose there exists a (f,j) such that $p_j > \sum_k D^f_{jk} \delta^f_k$ at a fixed point, then $x^{f,p}_j = x^p_{max}$ (from (6.3)). Claims 4.14 and 4.11 imply that $\sum_i x^i_j + \sum_f x^{f,r}_j \leq 1 + \sum_f x^{f,p}_j, \forall j$ and $\sum_j D^f_{jk} x^{f,p}_j \leq \sum_j C^f_{jk} x^{f,r} + T^f_k, \forall (f,k)$, which leads to a contradiction since $x^{f,p}_j = x^p_{max}$ cannot satisfy these constraints as discussed in the sufficiency conditions. This claim uses the *no production out of nothing* and *no vacuous production* conditions. □

Together Claims 4.11, 4.12, 4.14, 4.15 and 4.16 imply that none of the conditions (1.2), (2.2), (3.3), (4.3), (5.3), (6.3), and (7.3) are satisfied at fixed points of F. Therefore, we get the following theorem.

Theorem 4.17. *Assuming sufficient conditions of the existence of market equilibrium, every fixed point of F gives a solution of AD-NCP and hence a market equilibrium. Furthermore, F can be computed by a FIXP-circuit and hence market equilibrium computation problem for PLC utilities and PLC production is in FIXP.*

Remark 4.18. This technique can be used to obtain a Linear-FIXP (equivalent to PPAD) circuit for markets with SPLC utilities and SPLC production using the linear complementary problem (LCP) formulation given in [22], thereby giving an alternate proof of membership in PPAD for such markets.

Remark 4.19. The set of raw and produced goods being disjoint for each firm is not assumed in the Arrow-Debreu model [1]. We note that Theorem 4.17 easily extends to this case as well, because we do not restrict raw or produced goods to their respective sets since the beginning of this section.

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