IE598: Games, Markets, and Mathematical Programming

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1 Two Player Finite Simultaneous Games

A two player finite simultaneous game can be described by two payoff matrices A and B of size $m \times n$, where m and n are number of strategies of players 1 and 2 respectively. These strategies are called *pure* strategies. Let S_i denote the set of pure strategies of player i. Players 1 and 2 are also called *row* and *column* players respectively because player 1 can be thought of picking a row and player 2 can be thought of picking a column.

Example. Rock-Paper-Scissors: Each player has three strategies *rock, paper,* and *scissor*. The payoff matrices are:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Assumptions. We will assume that players are intelligent and rational, and they know the payoff matrices.

Definition 1.1 (Equilibria in pure strategies)**.** *A strategies pair $(i, j) \in S_1 \times S_2$ is said to be a pure strategy equilibrium iff i is a best response to j and j is a best response to i, i.e., i gives the best payoff that player 1 can get given that player 2 plays j and j gives the best payoff that player 2 can get given that player 1 plays i.*

Observe that there is no pure strategy equilibrium in the Rock-Paper-Scissor game. Since the existence of a pure equilibrium is not guaranteed, next we define a notion of *mixed strategy* where the players can randomize their pure strategies.

The set of mixed strategies of player 1 is defined as $\Delta(S_1) = \{x \in \mathcal{R}^n \mid 0 \leq x_i \leq 1, \forall i \in S_1; \sum_i x_i = 1\}$ and that of player 2 as $\Delta(S_2) = \{y \in \mathcal{R}^n \mid 0 \leq y_j \leq 1, \forall i \in S_2; \sum_j y_j = 1\}$. Note that mixed strategies contain all pure strategies. For convenience, we will denote $\Delta(S_1)$ as Δ^m since $|S_1| = m$, and $\Delta(S_2)$ as Δ^n.

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Observe that when the play is $(x, y) \in \Delta^m \times \Delta^n$, then the payoff of first player is $x^T Ay$ and that of second player is $x^T By$. For convenience, sometime we will ignore putting the transpose and write xAy instead of $x^T Ay$.

Definition 1.2 (Equilibria in mixed strategies)**.** *A strategies pair $(x, y) \in \Delta^m \times \Delta^n$ is said to be a mixed strategy equilibrium iff x is a best response to y and y is a best response to x, i.e.,*

$$xAy \geq \tilde{x}Ay, \ \forall \tilde{x} \in \Delta^m \text{ and } xBy \geq xB\tilde{y}, \ \forall \tilde{y} \in \Delta^n.$$

John Nash [15] proved in 1951 that every finite game has a mixed strategy equilibrium, which is also called a Nash equilibrium of the game. The proof uses a fixed point theorem and is highly non-constructive, i.e., it doesn't say anything about computing an equilibrium. The next question is the computation of equilibria.

1.1 Zero-Sum Games

For computation of an equilibrium, we first study an important and interesting subclass of two-player finite games called zero-sum games, where the sum of payoff matrices is a zero matrix, i.e., $A_{ij} + B_{ij} = 0, \forall i, j$ or $B = -A$. For example, Rock-Paper-Scissors is a zero-sum game.

Question. How to find an equilibrium in a zero-sum game?

For the answer to the above question, consider the following question: How much payoff the row player can guarantee itself? Note that if row player plays x then it will get at least $\min_{y \in \Delta^n} x^T Ay$ payoff. Let

$$\pi_1 = \max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T Ay \text{ and } x^* = \arg \max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T Ay.$$

Then, by playing x^* row player can guarantee itself a payoff of at least π_1.

Similarly, note that if column player plays y then it can ensure at least $-\max_{x \in \Delta^m} x^T Ay$ payoff or in other words, it can force at most $\max_{x \in \Delta^m} x^T Ay$ payoff on the row player. Let

$$\pi_2 = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T Ay \text{ and } y^* = \arg \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T Ay.$$

Then, by playing y^* column player can force a payoff of at most π_2 on the row player.

Observe that both players can compute (x^*, y^*) and (π_1, π_2) themselves using the payoff matrix A.

Theorem 1.3 (Minimax theorem (von Neumann and Morgenstern 1944))**.**

- $\pi_1 = \pi_2$.
- (x^*, y^*) *is a Nash equilibrium of game* $(A, -A)$.

Proof. Using the strategy profile (x^*, y^*), we get $\pi_1 \leq x^* A y \leq \pi_2$. This implies that $\pi_1 \leq \pi_2$. For the other direction (i.e., $\pi_1 \geq \pi_2$), we use the Nash' theorem that there exists a Nash equilibrium in every finite game. Suppose (x', y') be a Nash equilibrium of game $(A, -A)$. Then, we get $\pi_1 \geq x' A y' \geq pi_2$. □

Corollary 1.4. *All Nash equilibrium payoffs are same.*

Lemma 1.5. *If (x, y) and (x', y') are two Nash equilibria, then so are (x, y') and (x', y).*

Proof. Suppose x is not a best response to y'. Then

$$xAy \leq xAy' < x'Ay' \leq x'Ay \leq xAy,$$

which is a contradiction. □

Corollary 1.6. *The set of Nash equilibria is convex.*

Next, we design an alternate proof using the linear programming (LP) duality. For that, we ask the following question: Given the strategy y of column player, what is the maximum payoff row player can get?

Let A_i denote the i^{th} row of matrix A. Then, $A_i y$ is the payoff row player will get by playing pure strategy i. Therefore, $\max_{i \in S_1} A_i y$ is the maximum payoff row player can get for a pure strategy.

Claim 1.7. *Given y, the maximum payoff that row player can obtain from a mixed strategy is at most $\max_{i \in S_1} A_i y$.*

Lemma 1.8. *The following problems (1) and (2) are equivalent to each other.*

$$\pi_1 = \max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T A y. \tag{1}$$

$$\begin{array}{rcl} \max \pi_1 & & \\ \pi_1 & \leq & x^T A^j, \quad \forall j \in S_2 \\ \sum_i x_i & = & 1 \\ x_i & \geq & 0, \quad \forall i \in S_1 \end{array} \tag{2}$$

Similarly, we have the following theorem.

Lemma 1.9. *The following problems (3) and (4) are equivalent to each other.*

$$\pi_2 = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T A y. \tag{3}$$

$$\begin{array}{rcl} \max \pi_2 & & \\ A_i y & \leq & \pi_2, \quad \forall i \in S_1 \\ \sum_j y_j & = & 1 \\ y_j & \geq & 0, \quad \forall j \in S_2 \end{array} \tag{4}$$

Theorem 1.10 (Minimax theroem using LP duality)**.** *The LPs (2) and (4) are dual of each other. Hence, using the strong duality theorem of LPs, $\pi_1 = \pi_2$.*

Corollary 1.11. *Nash equilibria of a zero-sum game can be obtained by solving a LP, and hence is polynomial time computable.*

Remark 1.12. *Theorem 1.10 implies that the problem of computing Nash equilibria in a zero-sum game can be reduced to linear programming. [8, 1] gave reverse reduction by reducing LPs to zero-sum games. These results together imply that LPs and Zero-sum games are equivalent to each other.*

1.2 Non-zero-sum Games

In this section, we study general two-player game (A, B). First let's try to simplify the game. Given a game (A, B), if we construct another game (A', B) where $A'_{ij} = A_{ij} + c, \forall i, j$, i.e., a constant c is added to every entry of A. Then what is the relationship between the Nash equilibria (NE) of (A, B) and (A', B)?

Claim 1.13. *The set of Nash equilibria is same for games (A, B) and (A', B).*

Proof. Let (x, y) is a NE of game (A, B), then we have

$$x^T Ay \geq \tilde{x}^T Ay, \forall \tilde{x} \in \Delta^m \text{ and } x^T By \geq x^T B\tilde{y}, \forall \tilde{y} \in \Delta^n,$$

which is same as

$$x^T Ay + c \geq \tilde{x}^T Ay + c, \forall \tilde{x} \in \Delta^m \text{ and } x^T By \geq x^T B\tilde{y}, \forall \tilde{y} \in \Delta^n,$$

and hence

$$x^T A'y \geq \tilde{x}^T A'y, \forall \tilde{x} \in \Delta^m \text{ and } x^T By \geq x^T B\tilde{y}, \forall \tilde{y} \in \Delta^n.$$

The reverse direction also follows similarly, where a NE of (A', B) is also a NE of game (A, B). ☐

Now, suppose we consider two games (A, B) and (A', B') where $A'_{ij} = cA_{ij} + d, \forall i, j$ and $B'_{ij} = eB_{ij} + f, \forall i, j$, where $c, d > 0$ and e, f are any constants. Then, the following theorem is an easy extension of the above claim.

Theorem 1.14. *The set of Nash equilibrium of game (A, B) and (A', B') (as defined above) is same.*

Corollary 1.15. *For the purpose of computing Nash equilibria, we can without loss of generality assume that all entries of A and B are between 0 and 1, i.e., $0 < A_{ij}, B_{ij} < 1, \forall i, j$.*

Henceforth, we will assume that all entries of A, B are between 0 and 1.

Characterizing Nash equilibria. In this section, we will characterize the conditions for the strategies (x, y) to be a Nash equilibrium. Recall that when (x, y) is played, the expected payoff of player 1 and 2 are $x^T Ay$ and $x^T By$ respectively. Further, since x is a best response to y at a Nash equilibrium, if a pure strategy i is played with a non-zero probability then it should give the maximum payoff among all pure strategies, i.e.,

$$x_i > 0 \Rightarrow A_i y = \max_k A_k y, \forall i \in S_1$$

Let us introduce a variable π_1 that captures the maximum payoff that player 1 can get, i.e.,

$$x_i > 0 \Rightarrow A_i y = \max_k A_k y = \pi_1, \forall i \in S_1.$$

This implies that at a Nash equilibrium (x, y) we have

$$x_i \geq 0; \quad A_i y \leq \pi_1; \quad x_i(A_i y - \pi_1) = 0, \forall i \in S_1.$$

Similarly, for the second player, since y is a best response to x, we have

$$y_j \geq 0; \quad x^T B^j \leq \pi_2; \quad y_j(x^T B^j - \pi_2) = 0, \forall j \in S_2,$$

where π_2 captures the maximum payoff that player 2 can get. Putting these Together with the probability constraints give us

$$\begin{array}{ccc} y_j \geq 0, \ \ \forall j; & A_i y \leq \pi_1, \ \ \forall i; & \sum_j y_j = 1 \\ x^T B^j \leq \pi_2, \ \ \forall j; & x_i \geq 0, \ \ \forall i; & \sum_i x_i = 1 \\ y_j(x^T B^j - \pi_2) = 0; & x_i(A_i y - \pi_1) = 0. & \end{array} \tag{5}$$

The following lemma easily follows from the construction

Lemma 1.16. *A strategy profile (x, y) is a Nash equilibrium of game (A, B) if and only if it is a solution of* (5)*, where π_i captures the payoff of player i for* $i = 1, 2$.

Observe that the first two set of constraints are linear and hence easy to handle, the hard constraints are the last ones, which are quadratic. They are in fact a special kind of quadratic constraints, which pairs up a non-negativity constraint and a linear constraint and wants that either the variable is 0 or the corresponding linear constraint is tight (i.e., satisfies with the equality). This special quadratic formulation is called *Linear Complementarity Problem (LCP)* formulation.

So using Lemma 1.16 in order to find a Nash equilibrium, we need to solve the LCP given in (5). Next, we further simplify the LCP as follows. First, we simplify writing the above LCP by denoting the complementarity constraint with the $\perp$ symbol between the non-negativity constraint and the linear inequality constraint as follows:

$$\begin{array}{lcll} A_i y \leq \pi_1 & \perp & x_i \geq 0, & \forall i \in S_1 \\ x^T B^j \leq \pi_2 & \perp & y_j \geq 0, & \forall j \in S_2 \\ \sum_j y_j = 1; & & \sum_i x_i = 1 & \end{array} \tag{6}$$

Observe that the system (6) is same as the system (5). Next, our goal is to remove π_is from the above formulation, and for that we use Corollary 1.15 and assume that all entries of A, B are between 0 and 1. Using that, we can take π_is to the left hand side in (6) and we get:

$$\begin{array}{lcll} A_i(y/\pi_1) \leq 1 & \perp & x_i/\pi_2 \geq 0, & \forall i \in S_1 \\ (x/\pi_2)^T B^j \leq 1 & \perp & y_j/\pi_1 \geq 0, & \forall j \in S_2 \\ \sum_j y_j/\pi_1 = 1; & & \sum_i x_i/\pi_2 = 1 & \end{array} \tag{7}$$

Clearly, the system (7) is same as system (6). Next, we replace y_j/π_1 as $\tilde{y}_j$ and x_i/π_2 as $\tilde{x}_i$ and discard the last set of constraints and obtain the following:

$$\begin{array}{lcll} A_i\tilde{y} \leq 1 & \perp & \tilde{x}_i \geq 0, & \forall i \in S_1 \\ \tilde{x}^T B^j \leq 1 & \perp & \tilde{y}_j \geq 0, & \forall j \in S_2 \end{array} \tag{8}$$

The following lemma relates (8) and (7).

Lemma 1.17. *A solution (x, y, π_1, π_2) of (7) gives a solution of (8). Furthermore, a* **non-zero** *solution $(\tilde{x}, \tilde{y})$ of (8) gives a solution $(x, y, \pi_1, \pi_2) = (\frac{\tilde{x}}{\sum_i \tilde{x}_i}, \frac{\tilde{y}}{\sum_j \tilde{y}_j}, \frac{1}{\sum_j \tilde{y}_j}, \frac{1}{\sum_i \tilde{x}_i})$ of (7).*

It is easy to prove the above lemma. This lemma together with Lemma 1.16 implies that finding a Nash equilibrium of game (A, B) is equivalent to finding a non-zero solution of the LCP given by (8). In the next section, we will discuss the classic Lemke-Howson algorithm to find a non-zero solution of (8).

1.2.1 Lemke-Howson Algorithm

Before we begin with the Lemke-Howson algorithm, let us revise the basic facts about the polyhedra which will be useful in the algorithm.

A set of linear inequalities in n variables defines a polyhedron in n-dimension. A polytope is a bounded polyhedron. Each inequality defines a half-space and a tight inequality (i.e., satisfies with equality) is called a hyper-plane. We say that a polyhedron is non-degenerate if the number of hyper-planes (or tight inequalities) at a vertex is exactly n, and more generally the number of hyper-planes at a k-dimensional face of the polyhedron is exactly $n - k$. A vertex is a 0-dimensional face of the polyhedron, and an edge is a 1-dimensional face of the polyhedron. In a non-degenerate polyhedron of n-dimension, there are exactly n edges incident on a vertex, which can be obtained by relaxing a hyper-plane (i.e., not requiring the inequality corresponding to this hyperplane to be equality). Since there are n hyperplanes incident on a vertex, relaxing one will give one edge, and hence there are n edges incident on a vertex.

Observe that if we ignore the complementarity constraints (defined by $\perp$) in (8), then the set of linear inequalities defines a polytope. We say that the game (A, B) is non-degenerate if the polytope defined by the linear inequalities as in (8) (by ignoring the $\perp$ constraints) is non-degenerate. We remark that the degeneracy can be easily handled by several standard techniques, e.g., by perturbing the entries slightly. Henceforth, we will assume that the game is non-degenerate.

Recall that our goal is to find a non-zero solution of (8). Let's label the hyperplanes (corresponding to each inequality of (8)) as follows: Assign label i to $A_i y = 1$ and $x_i = 0$ for $1 \leq i \leq m$, and assign label $m + j$ to $x^T B^j = 1$ and $y_j = 0$ for $1 \leq j \leq n$. Note that there are a total of $m+n$ labels and each label is given to exactly two hyperplanes. Let P be the polytope defined by the linear inequalities in (8) (ignoring the $\perp$ constraints). Let v be a point in P and let $L(v)$ denote the union of labels of the inequalities tight at v (i.e., hyperplanes incident on v).

Definition 1.18. *A point v in polytope P is called completely labeled if all labels are present at v, i.e., $L(v) = \{1, 2, \ldots, m + n\}$.*

What are the solutions of (8)? If (x, y) is a solution of (8) then it has to satisfy all linear inequalities (i.e., a feasible point in P) and all complementary (the $\perp$ constraints). Note that it is easy to obtain a feasible point in P because that is equivalent to solving a linear program. The problem here is that we want a feasible point which also satisfies all the complementarity constraints. Furthermore, since there are $m + n$ complementarity constraints (i.e., at a solution at least $m+n$ inequalities must be tight) and the dimension of polytope P is $m + n$, every solution of (8) is at a vertex of P. Since if a feasible point of P satisfies all complementarity constraints, then it should have all the labels because each complementarity has a different label. From this discussion, we can conclude that every solution of (8) is a completely labeled vertex of P and vice-versa. Therefore, finding a non-zero solution of (8) reduces to finding a non-zero completely labeled vertex of P.

Now, observe that $(x, y) = (0, 0)$ is both feasible (a vertex of P) and completely-labeled, hence it is a trivial solution of (8). The Lemke-Howson algorithm starts with the trivial completely-labeled vertex $(0, 0)$ and fix a label k, $1 \leq k \leq m+n$, and then it relaxes the hyperplane corresponding to the label k. Since now we have exactly $m + n - 1$ inequalities tight, we are on an edge. Then it keeps moving on this edge defined by the $m + n - 1$ tight inequalities until it hits another hyperplane (a new inequality becomes just tight). At this new vertex, either it is completely-labeled (in that case we are done) or there has to be some duplicate label (because there are exactly $m + n$ hyperplanes incident on any vertex and there are $m + n$ distinct labels, which implies that if not all labels are present then some label must be duplicate. Further, observe that we had all labels present except k on the edge, so when we hit a new hyperplane then either we acquire k or some $j \neq k$. In the latter case, we have exactly one duplicate label, that is, j.

Now since there are two hyperplanes corresponding to label j, one is recently acquired, the algorithm relaxes the other hyperplane corresponding to label j, and we again move on an edge of P, where all labels are present except k. By doing this, we again hit another hyperplane and the same situation repeats. We claim that this process has to stop at a non-zero completely-labeled vertex.

For the correctness, we need to show that: The process of moving from one vertex to another always visits a new vertex and since the number of vertices are finite, the process has to stop at a non-zero completely labeled vertex.

For the contradiction, suppose the algorithm comes back to the previously visited vertex, then there are two cases: First, it comes back to $(0,0)$, in that case since $(0,0)$ is completely-labeled and the entire path has all the labels except k, there is only one way to move from $(0,0)$ such that all labels are present except k, which is to relax the hyperplane corresponding to the label k. For the another case, suppose it comes back to an intermediate vertex, in that case since this intermediate vertex has exactly one duplicate label, say j, and there are exactly two ways to move from this vertex such that all labels are present except k, which is to relax one of the hyperplanes corresponding to label j. And if the process comes back to this vertex, then it implies that there are 3 ways to move from this vertex, which is a contradiction.

In summary, the Lemke-Howson algorithm finds a Nash equilibrium (in fact, it shows that there exists one without using fixed point theorems), but in the worst case it may have to go over an exponential number of vertices. However, the algorithm in general works quite fast in practice. Also, as corollaries, we can deduce from the Lemke-Howson algorithm that all Nash equilibria are rational if all entries of A and B are rational valued numbers, and there are odd number of equilibria.

1.2.2 Support Enumeration Algorithm

In the last section, we saw that the Lemke-Howson algorithm converges to a Nash equilibrium but it doesn't give any guarantee about it whether it is the only equilibrium or whether it is the best in terms of the total expected payoff, etc. Furthermore, in some applications we desire all Nash equilibria or the one with a particular feature. In this section, we answer these questions using the support enumeration algorithm.

A support of a mixed strategy, say $x = (x_1, \ldots, x_m)$, is defined as

$$supp(x) = \{i \mid x_i > 0\},$$

i.e., the set of pure strategies that are played with non-zero probability. For example, if $x = (0.5, 0, 0.2, 0.3, 0)$, then $supp(x) = \{1, 3, 4\}$.

Recall that we have discussed in the previous sections that finding a Nash equilibrium is a hard problem. Suppose we know the support of an equilibrium (but not the actual probabilities), does finding the Nash equilibrium corresponding to the given support becomes easy?

Given $supp(x)$ and $supp(y)$, we know that (x, y) needs to satisfy the constraints that each pure strategy in the support must give the maximum payoff, i.e.,

$$\begin{array}{ll} A_i y = \pi_1, & \forall i \in supp(x) \\ A_i y \leq \pi_1, & \forall i \notin supp(x) \\ x^T B^j = \pi_2, & \forall j \in supp(y) \\ x^T B^j \leq \pi_2, & \forall j \notin supp(y) \\ \sum_i x_i = 1 & \\ \sum_j y_j = 1 & \\ x_i \geq 0, & \forall i \in S_1 \\ y_j \geq 0, & \forall j \in S_2 \\ x_i = 0, & \forall i \notin supp(x) \\ y_j = 0, & \forall j \notin supp(y) \end{array}, \qquad (9)$$

where π_1 and π_2 captures the payoffs of player 1 and 2 respectively at the Nash equilibrium whose support is $supp(x)$ and $supp(y)$. The first four constraints capture that each pure strategy in the support must give the maximum payoff to both the players, and the remaining constraints capture the probability distribution whose support is $supp(x)$ and $supp(y)$.

Notice that the (9) is a system of linear inequalities which can be solved efficiently (in polynomial time) using linear programming. Verify that any feasible point, satisfying (9) is a Nash equilibrium of the game (A, B). However, if the feasible region is empty, then that implies that there is no Nash equilibrium with the given support pair.

Thus, the problem of computing a Nash equilibrium becomes easy if we know the support of equilibrium. However, we don't know the support and in that case we can enumerate over all support pairs and that will not only find one equilibrium but all equilibria in a non-degenerate game. There are 2^{m+n} support pairs when A and B are of $m \times n$ dimension, which is exponential in the number of strategies. And for each support pair we need to solve one linear feasibility program as in (9) that can be done in polynomial time using the ellipsoid or interior point method.

2 Correlated Equilibria

We noticed that there might be many Nash equilibria so it is not clear which one will be played. Furthermore, even if the Nash equilibrium is unique, the payoff can be very bad (e.g., prisoner's dilemma). To address these issues, next we consider a more general solution concept called correlated equilibrium, first discussed by Robert Aumann in 1974.

Let p_{ij} denote the probability by which the strategy profile (i, j) is played at a Nash equilibrium, where player 1 plays the i^{th} strategy and player 2 plays the j^{th} strategy and it can be denoted by

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}.$$

Example 2.1. *Consider the game of chicken given by the following payoffs:*

$$\begin{bmatrix} 0,0 & 5,1 \\ 1,5 & 4,4 \end{bmatrix}$$

It's a symmetric game and let the two strategies are (S, C). There are two pure Nash equilibria of this game given by (S, C) and (C, S), and a mixed Nash equilibrium given by $((1/2S + 1/2C), (1/2S + 1/2C))$, i.e., play S and C with 1/2 probability each.

The probability distribution p_{ij}s at Nash equilibria of this game is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}.$$

In Nash equilibria, players choose the strategies independently. In contrast, suppose there is a third trusted party or a mechanical device (called coordinator) who draws a sample from a public (known to both the players) distribution $[p_{ij}]_{i \in S_1, j \in S_2}$ before the game is played, and suppose the strategy profile (i, j) is drawn then the coordinator advises the first player to play the i^{th} strategy and advises the second player to play the j^{th} strategy, privately (i.e., players don't know what is the advice given to the other player). Then the distributions p_{ij}s under which if it is best for each player to follow the advice assuming that the other player is going to follow the advice, then this probability distribution is called a correlated equilibrium. The set of correlated equilibria are the set of such probability distributions.

Suppose $[p_{ij}]_{i \in S_1, j \in S_2}$ is a correlated equilibrium. Then, with probability p_{ij} the strategy profile (i, j) will be drawn. Since this is a correlated equilibrium, it is best for the first player to follow the advice assuming that the second player follows the advice given to him/her. This implies that

The expected payoff of a player by following the advice should be at least the expected payoff from not following the advice (assuming that the other player follows the advice given to him), i.e., for the first player we have,

$$\sum_j A_{ij} \frac{p_{ij}}{\sum_k p_{ik}} \geq \sum_j A_{i'j} \frac{p_{ij}}{\sum_k p_{ik}}, \quad \forall i, i' \in S_1,$$

where $\frac{p_{ij}}{\sum_k p_{ik}}$ is the probability that the second player is advised to play j given that the first player is advised to play i. Similarly, for the second player we have

$$\sum_i B_{ij} \frac{p_{ij}}{\sum_k p_{kj}} \geq \sum_i B_{ij'} \frac{p_{ij}}{\sum_k p_{kj}}, \quad \forall j, j' \in S_2.$$

By simplifying above, and adding the probability distribution constraint, the set of correlated equilibria are given by:

$$\begin{array}{ll} \sum_j A_{ij} p_{ij} \geq \sum_j A_{i'j} p_{ij}, & \forall i, i' \in S_1 \\ \sum_i B_{ij} p_{ij} \geq \sum_i B_{ij'} p_{ij}, & \forall j, j' \in S_2 \\ \sum_{i,j} p_{ij} = 1 & \\ p_{ij} \geq 0, & \forall i, j \end{array} \tag{10}$$

Let NE and CE denote the set of Nash equilibria and correlated equilibria respectively. The following claim is easy to check.

Claim 2.2. *$NE \subseteq CE$.*

The above claim implies using the Nash's theorem that the set of correlated equilibria is non-empty. In Example 2.1, the expected total payoff at the three Nash equilibria are $6, 6$, and 5 respectively. There are many correlated equilibria in this example: verify that one which has the maximum expected total payoff of $6^2/_3$ is given by the distribution

$$\begin{bmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}.$$

Question: Why the above distribution is not possible at a Nash equilibrium?

3 Cooperative Game Theory

3.1 Nash Bargaining and Cooperation in 2-player Game

The problem with the concept of correlated equilibria is that there are many equilibria and hence there would be conflict of interest about selecting one particular equilibrium. We need a theory of *cooperative equilibrium selection*.

Let's begin with an example.

Example 3.1. Divide the dollar game (2 players). *Both players simultaneously propose an allocation to divide $100. They get their allocation if they propose the same allocation, otherwise each gets* $0.

Observe that there are many equilibria in this game. Suppose players don't know each other, but want to win some money. What would you do if you play this game? Why?

Focal Point (or Focal equilibrium) (in absence of communication) When there are multiple equilibria, there is one which can be determined by any of a wide range of factors including environment, culture, etc. For example, one which is impartial, fair, natural. To quote Thomas Schelling: "It is each person's expectation of what the other expects him/her to expect to be expected to do."

The question is: What is the reasonable bargaining solution? Let's first define the problem formally. The two-player bargaining problem is defined by (F, v) where F is a closed convex set of $\mathcal{R}^2$ denoting a set of feasible payoffs, $(v_1, v_2) \in \mathcal{R}^2$ is the disagreement point, and the set $F \cap \{(x_1, x_2) \mid x_1 \geq v_1; x_2 \geq v_2\}$ is non-empty and bounded.

First of all, given a two player game, how would you get (F, v)? F can be obtained as the set of payoff tuple at a correlated equilibrium of the game, and v_i can be minimax payoff of player $i = 1, 2$ which player i can ensure himself/herself without cooperating with the other player.

The goal is to find a solution function $\phi : (F, v) \to (x_1, x_2) \in \mathcal{R}^2$, where x_i is the payoff allocation of player $i = 1, 2$. What are the reasonable properties ϕ should satisfy? Nash approached this problem axiomatically.

Axiom 1. *Strong efficiency* or Pareto efficiency. If $\phi(F, v) = (x_1, x_2)$ then there shouldn't exist another feasible point $(x'_1, x'_2) \in F$ such that $(x'_1, x'_2) \neq (x_1, x_2)$ and $x'_i \geq x_i$ for $i = 1, 2$. An inefficient outcome is unlikely due to the space for renegotiation.

Axiom 2. *Individually rational.* $\phi(F, v) \geq v$, i.e., $\phi_i(F, v) \geq v_i, \forall i = 1, 2$.

Axiom 3. *Invariance to equivalent payoff representations* or change the way we measure the utility/payoff. For any number $\lambda_1 > 0, \lambda_2 > 0, \gamma_1, \gamma_2$ such that

$$G = \{\lambda_1 x_1 + \gamma_1, \lambda_2 x_2 + \gamma_2 \mid (x_1, x_2) \in F\} \text{ and } w = (\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2),$$

Then $\phi(G, w) = (\lambda_1 \phi_1(F, v) + \gamma_1, \lambda_2 \phi_2(F, v) + \gamma_2)$.

Axiom 4. *Independence of irrelevant alternatives.* If $G \subseteq F$ (G is closed and convex) and $\phi(F, v) \in G$ then $\phi(G, v) = \phi(F, v)$.

Axiom 5. *Symmetry.* If $\{(x_2, x_1) \mid (x_1, x_2) \in F\} = F$ and $v_1 = v_2$ then $\phi_1(F, v) = \phi_2(F, v)$.

Theorem 3.2 (Nash bargaining solution). *There is exactly one function $\phi(.,.)$ that satisfies all axioms,*

$$\phi(F, v) = \arg\max_{x \in F, x_i \geq v_i} (x_1 - v_1)(x_2 - v_2).$$

The nice Properties of the Nash bargaining solution is that it is unique and it always exists.

What is the Nash bargaining solution for divide the dollar game? Check that it is $(50, 50)$.

What happens for more than 2 players? The Nash bargaining solution for n players is:

$$\phi(F, v) = \arg\max_{x \in F, x_i \geq v_i} \prod_{i=1}^{n} (x_i - v_i).$$

We next consider the variants of divide the dollar game to understand how good the Nash bargaining solution for $n > 2$ players.

Example 3.3. *Consider the variant of divide the dollar game. Let $n = 3$ and total money is \$300. Players get 0 unless all three propose the same non-negative allocation and total must be at most 300, in which case they get this allocation.*

Observe that $F = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leq 300; x_i \geq 0, \forall i\}$ and $v = (v_1, v_2, v_3) = (0, 0, 0)$. It is easy to check that the Nash bargaining solution is $(100, 100, 100)$.

Example 3.4. *Consider another variant of divide the dollar game. Let* $n = 3$ *and total money is* \$300*. Players get* 0 *unless players 1 and 2 propose the same non-negative allocation and the total must be at most* 300 *in which case they get this allocation.*

Observe that $F = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leq 300; x_i \geq 0, \forall i\}$ *and* $v = (v_1, v_2, v_3) = (0, 0, 0)$*. It is easy to check that the Nash bargaining solution is again* $(100, 100, 100)$*.*

Observe that the Nash bargaining solution is reasonable for the first example but unreasonable for the second. The problem with the Nash bargaining solution when there are more than 2 players it that it completely ignores the possibility of cooperation among subsets of players. So it is not widely used for the analysis of game with more than 2 players.

Consider One more example before we proceed with another solution concept for multi players.

Example 3.5. *Consider one more variant of divide the dollar game. Let* $n = 3$ *and total money is* \$300*. Players get* 0 *unless* any *two players propose the same non-negative allocation and the total must be at most* 300 *in which case they get this allocation.*

Observe that $F = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leq 300; x_i \geq 0, \forall i\}$ *and* $v = (v_1, v_2, v_3) = (0, 0, 0)$*. It is easy to check that the Nash bargaining solution is again* $(100, 100, 100)$*, but how different is this example from the previous ones.*

The above example is very complicated to analyze due to interaction among different subset of players. We will need a notion of coalition and transferable utility. Any non-empty subset of players is called a *coalition*, and a coalition with all players is called *grand coalition*.

Transferable utility. A common commodity – money – that players can freely transfer among themselves.

Now we can assign a number to each coalition.

$$v : 2^N \to \mathcal{R},$$

where N is the set of players, and $v(S)$ is the worth of coalition S, i.e., total amount of transferable utility S can earn without any help from outside players. Clearly, $v(\emptyset) = 0$. v represents a coalition game. The worth of each coalition in the above examples are as follows:

Example 1: $v(\{1,2,3\}) = 300, v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$.

Example 2: $v(\{1,2,3\}) = v(\{1,2\}) = 300$, $v(\{1,3\}) = v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$.

Example 3: $v(\{1,2,3\}) = v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 300, v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$.

3.2 The Core

An allocation x is in core of a coalition game v if

$$\sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \text{ and } \sum_{i \in N} x_i = v(N).$$

If an allocation is not in the core, then there is some coalition S such that the players in S could all do strictly better than in x by cooperating together and dividing the worth $v(S)$ among themselves.

The core of Example 1: $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 300\}$
The core of Example 2: $\{(x_1, x_2, 0) \mid x_1 + x_2 = 300\}$
The core of Example 3: empty

The problem with the concept of the core is it can be empty or very large. That makes it difficult as a predictive theory. What we want is a theory that predicts, for each game in coalitional form, a unique expected payoff allocation for the players. Or in other words, we would like to identify some mapping $\psi : (N, v) \to \mathcal{R}^N$, where ψ_i is the payoff of player i.

3.3 Shapley Value

Shapley approached the above problem axiomatically, i.e., what kind of properties we might expect $\psi : (N, v) \to \mathcal{R}^N$ to satisfy?

Axiom 1. *Symmetry.* If players i and j contribute the same amount to each coalition of other players then they should get the same allocation, i.e., if $v(S \cup \{i\}) = v(S \cup \{j\}), \forall S \subseteq N \setminus \{i, j\}$, then $\psi_i(N, v) = \psi_j(N, v)$.

Axiom 2. *Dummy Player.* If player i contributes to any coalition of other players is exactly that i is able to achieve alone, then i should get exactly that, i.e., if $v(S \cup \{i\}) - v(S) = v(\{i\})$ then $\psi_i(N, v) = v(\{i\})$.

Axiom 3. *Additivity* Two coalition games v_1 and v_2 involving the same set of players. Suppose we remodel the setting as a single game in which each coalition S receives a payoff of $v_1(S) + v_2(S)$, then $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$.

Theorem 3.6 (Shapley Value Theorem)**.** *There is a unique function ψ that satisfies all three axioms and it is given by:*

$$\psi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)).$$

The above formula can be interpreted by considering a queue of all players outside a room in which exactly one player can enter at a time. Suppose players randomly line up in a queue. Note that there are $|N|!$ different ways in which the players can be lined up in a queue. Further, for any set S of players that doesn't contain player i there are exactly $|S|!(|N| - |S| - 1)!$ different ways to order the players such that S is the set of players who are ahead of i in the queue. Hence, with $\frac{|S|!(|N|-|S|-1)!}{|N|!}$ probability, When player i enters the room, he/she will find S set of players already in the room. In that case, player i marginal contribution to the worth of coalition in the room when he enters is $v(S \cup \{i\}) - v(S)$. Hence, the Shapley value of any player is his/her expected marginal contribution when he/she enters the room.

We remark that Shapley value is a powerful tool to evaluate the power structure of the coalition game. Compute the Shapley value in above three examples.

Example (Voting Game). A parliament of country ABCD is made of 4 political parties A, B, C, D, and they have 45, 25, 15, 15 representatives respectively. They need to vote for a 100 million dollar ($100M) spending bill whether to pass it or not, and how much of this amount is controlled by each party. A majority (minimum of 51) votes are needed to pass the legislation. If the bill doesn't pass then each party gets 0 to spend.

What are the winning coalitions of this game? What is the coalition game v?

$$v(\{A,B\}) = 100M = v(\{A,B,C\}) = ...$$

$$v(\{A\} = 0 = v(\{B\}) = ...$$

Check that the Shapley value of A is $\$50M$ and players B, C, D are symmetric and each gets $\$50/3M$ each. Note that even though B has more representatives in the parliament compared to C and D, but it has the same power (in case of majority voting) like C and D. Further, check that the core of this coalition game is empty.

4 Linear Complementarity Problem and Lemke's Algorithm

In this section, we define linear complementarity problem (LCP) and the Lemke's algorithm for solving it.

Definition 4.1. *Given a $n \times n$ matrix M and a $n \times 1$ vector q, find y such that the following is satisfied:*

$$My \leq q; \quad y \geq 0; \quad y^T(My - q) = 0 \ . \tag{11}$$

Note that the first constraint is $M_i y \leq q_i$, for $1 \leq i \leq n$, the second constraint is non-negative constraint, i.e., $y_i \geq 0, \forall i$. These two set of constraints are linear, but the last constraint is a quadratic constraint, i.e., $\sum_i y_i(M_i y - q_i) = 0$. The last constraint can be simplified to $y_i(M_i y - q_i) = 0, \forall i$ because $M_i y - q_i \leq 0$ and $y_i \geq 0, \forall i$.

The (4.1) can be written as where the $\perp$ denote that there is a complementarity condition between the two linear inequalities.

$$M_i y \leq q_i \quad \perp \quad y_i \geq 0, \quad \forall i \ . \tag{12}$$

Now the question is how to solve (12). Observe that if $q_i \geq 0, \forall i$, then $y_i = 0, \forall i$ is a trivial solution. However, it is not clear how to find a solution if some q_is are negative. Also, observe that it is not clear how to use Lemke-Howson algorithm on this LCP because we don't have a starting completely labeled vertex.

Note that the number of variables in (12) is n and we need at least n linear inequalities to be tight (to satisfy the complementarity constraints) at a solution. It implies that every solution of (12) is at a vertex of the polyhedron defined by the linear constraints.

4.1 Lemke's Algorithm

To solve (12), Lemke [14] added an auxiliary non-negative variable z and considered the following LCP:

$$M_i y - z \leq q_i \quad \perp \quad y_i \geq 0, \quad \forall i; \quad z \geq 0 \ . \tag{13}$$

Note that every solution y of (12) gives a solution $(y, 0)$ of (13), and every solution (y, z) of (12) with $z = 0$ gives a solution y of (12). This implies that finding a solution of (12) is equivalent to finding a solution of (13) with $z = 0$.

Furthermore, the number of variables in (13) is $n + 1$, however we still have n complementarity constraints. Therefore, every solution of (13) is either at a vertex or on an edge of the polyhedron defined by the linear constraints. Let us assign label i to $M_i y - q_i \leq 0$ and $y_i \geq 0$. Let S denote the set of solutions of (13). Consider a vertex solution v of S, there are $n+1$ linear inequalities tight at v. Since v is a solution and hence it satisfies all the complementarity constraints, there are n linear inequalities corresponding to n different labels. The one extra linear tight inequality comes from either $z = 0$ or some duplicate label k such that $M_k y - z = q_k$ and $y_k = 0$ at v.

Suppose we want to move away from v and still in S, then in the former case the only possible way is to relax the tight inequality corresponding to $z = 0$ and in the latter case there are two possible ways one relaxing $M_k y - z = q_k$ and another by relaxing $y_k = 0$. These together implies that there is one edge incident on v if it has $z = 0$, otherwise there are two edges incident on v.

The above discussion implies that S consists of a set of paths, whose endpoints are either vertices with $z = 0$ or infinite edges (also called rays). And our goal is to find a vertex with $z = 0$. The advantage of working with (13) is that if we plug in $y = 0$ then we get a ray (called *primary*)starting from $(y = 0, z = \infty)$ to $(y = 0, z = (\max_k\{-q_k\}, 0)$, which is in S. When we reach at the vertex corresponding to $(y = 0, z = (\max_k\{y_k\}, 0)$, then this vertex has either $z = 0$ (in that case we are done) or there is a duplicate label and using that we can pivot (like in the Lemke-Howson (LH) algorithm). We repeat this procedure until we find a vertex with $z = 0$.

Like in the LH algorithm, we can easily show that the path followed by the Lemke's algorithm never revisits another vertex, but unlike the LH algorithm the set of linear Inequalities now define a polyhedron (instead of polytope in case of Nash equilibrium problem) and hence there is a danger that the Lemke's algorithm may converge on a ray (called secondary).

In summary, Lemke's algorithm works on any (13) but may converge on a secondary ray even though there is a solution of (12). This is clearly unavoidable when the (12) has no solution. In order to achieve guaranteed convergence to a solution, the proof of no secondary rays is needed, which states that there are no secondary rays in the solution set S of (13).

5 Linear Markets

In this section, we define two fundamental market models: Fisher and Exchange. The Fisher model was defined by Irving Fisher in 1891 [4] and Exchange in [19, 2].

5.1 The Fisher Market Model

In a Fisher market, there is a set A of n buyers (or agents) and a set G of m *divisible* goods. Each buyer i comes with a budget of M_i dollars and has a linear utility function over bundle of goods. The utility of buyer i from a bundle $x_i = (x_{ij})_{j \in G}$ is given by $\sum_{j \in G} U_{ij} x_{ij}$, where U_{ij} is the utility from one unit of good j. Note that U_{ij}s define the utility function of buyers. At given prices $p = (p_1, \ldots, p_m)$, where p_j is price per unit of good j, each buyer buys a utility maximizing (optimal) bundle of goods subject to budget constraints, i.e., a bundle $x_i = (x_{i1}, \ldots, x_{in})$ such that

$$\max \sum_{j \in G} U_{ij} x_{ij} \text{ subject to } \sum_{j \in G} x_{ij} p_j \leq M_i.$$

At equilibrium prices p, each buyer gets an optimal bundle and market clears (demand meets supply), i.e., $\sum_i x_{ij} = s_j, \forall j \in G$, where s_j is the total supply of good j. We can assume without loss of generality that $s_j = 1, \forall j$ by scaling the U_{ij}s appropriately. Henceforth, we will assume that each good comes in unit supply.

Without loss of generality, we will also assume that each buyer i is interested in at least one good, i.e., $U_{ij} > 0$ for some $j \in G$. If for some buyer i, $U_{ij} = 0, \forall j \in G$ then we can discard this buyer from the market. Similarly, we will assume that for each good $j \in G$, there is at least one buyer who is interested in it, i.e., for each j, there exists a buyer i such that $U_{ij} > 0$. If for some good j, $U_{ij} = 0, \forall i$, then we can discard this good from the market.

The linear Fisher market equilibrium problem is to find equilibrium prices and allocation for a given market.

Example 5.1. *Consider a market with 2 buyers and 2 goods. The budget of buyer 1 and 2 are \$10 each, and the utility functions are:* $U_{11} = 1, U_{12} =$

$0, U_{21} = 0, U_{22} = 1$. *What are the equilibrium prices and allocation in this market?*

Check that $p_1 = p_2 = 10$ *and* $x_{11} = x_{22} = 1, x_{12} = x_{21} = 0$ *is the only equilibrium in this market.*

Example 5.2. *Let's modify the above market. The budget of buyer 1 and 2 are \$10 each, and the utility functions are:* $U_{11} = 1, U_{12} = 0, U_{21} = 2, U_{22} = 1$. *What are the equilibrium prices and allocation in this market?*

Check that $p_1 = 40/3, p_2 = 20/3$ *and* $x_{11} = 3/4, x_{21} = 1/4, x_{21} = 0, x_{22} = 1$ *is the only equilibrium in this market.*

Question: Can price of some good j be zero at an equilibrium? No, since under our assumption there is a buyer i such that $U_{ij} > 0$, so if $p_j = 0$, then i will demand for an infinite amount of good j which will then not satisfy the market clearing constraints. Hence, we can deduce that equilibrium prices of all goods are non-zero.

5.2 The Exchange Market Model

In an exchange market, there is a set A of n agents and a set G of m *divisible* goods. Each agent i comes to market with an endowment of $W_i = (W_{i1}, \ldots, W_{im})$, where W_{ij} is the amount of good j, and it has a linear utility function over bundle of goods. The utility of agent i from a bundle $x_i = (x_{ij})_{j \in G}$ is given by $\sum_{j \in G} U_{ij} x_{ij}$, where U_{ij} is the utility from one unit of good j. At given prices $p = (p_1, \ldots, p_m)$, where p_j is price per unit of good j, each agent first earns money by selling its endowment and then buys a utility maximizing (optimal) bundle of goods subject to budget constraints, i.e., a bundle $x_i = (x_{i1}, \ldots, x_{in})$ such that

$$\max \sum_{j \in G} U_{ij} x_{ij} \text{ subject to } \sum_{j \in G} x_{ij} p_j \leq \sum_{j \in G} W_{ij} p_j.$$

At equilibrium prices p, each agent gets an optimal bundle and market clears (demand meets supply), i.e., $\sum_i x_{ij} = \sum_i W_{ij}, \forall j \in G$. We can assume without loss of generality that $\sum_i W_{ij} = 1, \forall j$ by scaling the U_{ij}s appropriately. Henceforth, we will assume that each good comes in unit supply.

The following claim is easy to check.

Claim 5.3. *If* p *are equilibrium prices of an exchange market, then so are* $\alpha.p, \forall \alpha > 0$.

Observe that the difference between Fisher and exchange models is only in the money. In Fisher, it is fixed in the input, whereas in exchange it depends on the prices of the goods.

Fisher model is a special case of exchange market model, where a given Fisher market $\mathcal{M}$ with input parameters (M_i, U^f_{ij}) can be reduced to an exchange market $\mathcal{M}'$ with input parameters (W_{ij}, U^e_{ij}), where $W_{ij} = M_i / \sum_k M_k, \forall i, j$ and $U^e_{ij} = U^f_{ij}, \forall i, j$. Check that an equilibrium (p, x) of $\mathcal{M}'$ gives an equilibrium $(p \frac{\sum_k M_k}{\sum_j p_j}, x)$ of $\mathcal{M}$.

5.2.1 Equilibrium Characterization

Let's first capture the optimal bundles of agents. Given prices p, what is the optimal bundle x_i of agent i? Note that U_{ij}/p_j is the utility per unit of money from good j (also called bang-per-buck). The maximum utility can be obtained by purchasing only those goods which gives the maximum utility per unit of money (or maximum bang-per-buck (MBB)). Furthermore, if there are two goods j and j' which are MBB goods for buyer i, then i is indifferent in both these goods because both goods gives the same utility per unit of money.

From these observations, we can conclude that the optimal bundle x_i of agent i must consists of only MBB goods, i.e.,

$$x_{ij} > 0 \Rightarrow \frac{U_{ij}}{p_j} = \max_{k \in G} \frac{U_{ik}}{p_k}, \forall j \in G.$$

Let's capture the inverse of maximum bang-per-buck in a variable λ_i. Then the optimal bundle constraint is:

$$x_{ij} > 0 \Rightarrow \frac{U_{ij}}{p_j} = \max_{k \in G} \frac{U_{ik}}{p_k} = \frac{1}{\lambda_i}, \forall j \in G.$$

This gives us the following complementarity constraints that capture the optimal bundles for each buyer i:

$$x_{ij} \geq 0 \quad \perp \quad \tfrac{U_{ij}}{p_j} \leq \tfrac{1}{\lambda_i}, \quad \forall i \in A, j \in G \ ,$$

which is after simplifying

$$x_{ij} \geq 0 \quad \perp \quad U_{ij}\lambda_i \leq p_j, \quad \forall i \in A, j \in G \ . \tag{14}$$

Next, let's capture the market clearing constraints, i.e.,

$$\begin{array}{l} \sum_{j \in G} x_{ij} p_j = \sum_j W_{ij} p_j, \quad \forall i \in A \\ \sum_{i \in A} x_{ij} = \sum_i W_{ij} = 1, \quad \forall j \in G \end{array} . \tag{15}$$

(14) and (15), together, characterize market equilibria of a linear exchange market provided that $p > 0$.

$$p_j > 0, \quad \forall j \in G \ . \tag{16}$$

The proof of the following lemma easily follows from the above discussion.

Lemma 5.4. *(p, x) is a market equilibrium of a linear exchange market if and only if they are solution of* (14), (15) *and* (16).

5.2.2 LCP Formulation

In this section, we derive a LCP formulation for the linear exchange markets. Lemma 5.4 states that (14), (15) and (16) captures the set of equilibria however these constraints don't define an LCP because (i) the first constraint of (15) is not even linear, (ii) all variables don't have complementarity constraints, and (iii) the (16) is a strict inequality. Next, we fix all these issues. For the first, let us replace x_{ij} by f_{ij}/p_j where f_{ij} denotes the money spent by agent i on good j. Note that given f_{ij}'s and p_j's, we can easily get x_{ij}. We get the following:

$$\begin{array}{ccl} f_{ij} \geq 0 & \perp \quad U_{ij}\lambda_i \leq p_j, & \forall i \in A, j \in G \\ \sum_{i\in A} f_{ij} = p_j, & & \forall j \in G \\ \sum_{j\in G} f_{ij} = \sum_j W_{ij}p_j, & & \forall i \in A \\ p_j > 0, & & \forall j \in G \end{array} \quad . \tag{17}$$

Now, we replace the second and third constraints of (17) as $-\sum_{j\in G} f_{ij} \leq -\sum_j W_{ij}p_j$ and $\sum_i f_{ij} \leq p_j$. Note that if we sum these two constraints for all i and all j respectively, then we get

$$\sum_j p_j = \sum_j p_j \sum_i W_{ij} = \sum_{i,j} W_{ij}p_j \leq \sum_{i,j} f_{ij} \leq \sum_j p_j \ ,$$

which implies that all inequalities must be strict. And hence these inequalities are equivalent to the equalities. We can further have complementarity constraints with these inequalities which are essentially redundant, and we get

$$\begin{array}{lccl} f_{ij} \geq 0 & \perp & U_{ij}\lambda_i \leq p_j, & \forall i \in A, j \in G \\ p_j \geq 0 & \perp & \sum_{i\in A} f_{ij} \leq p_j, & \forall j \in G \\ \lambda_i \geq 0 & \perp & -\sum_{j\in G} f_{ij} \leq -\sum_j W_{ij}p_j, & \forall i \in A \\ p_j > 0, & & & \forall j \in G \end{array} \quad . \tag{18}$$

Finally, to take care of the last issue of the last inequality being strict, recall Claim 5.3 that if p are equilibrium prices then so are $\alpha p, \forall \alpha > 0$, we replace $p_j = p'_j + 1$, i.e., we enforce that each $p_j \geq 1$. That takes care of the strict inequality. Finally, we get

$$\begin{array}{lccl} f_{ij} \geq 0 & \perp & U_{ij}\lambda_i \leq p'_j + 1, & \forall i \in A, j \in G \\ p'_j \geq 0 & \perp & \sum_{i\in A} f_{ij} \leq p'_j + 1, & \forall j \in G \\ \lambda_i \geq 0 & \perp & -\sum_{j\in G} f_{ij} \leq -\sum_j W_{ij}(p'_j + 1), & \forall i \in A \end{array} \quad . \tag{19}$$

The following theorem is straightforward based on the above construction:

Theorem 5.5. *Every market equilibrium of a linear exchange market is a solution of the* (19) *and vice-versa, up to scaling.*

The LCP in (19) is in a standard form, however there is no starting solution to start with. So to apply Lemke's algorithm, we need to add an auxiliary dimension z, and for that we only add z to the third constraint which not only gives us a starting solution of the modified LCP but also there is a economic meaning of z which is the surplus of agent i (i.e., earning - spending). So, we apply the Lemke's algorithm on the following LCP:

$$\begin{array}{rcll} f_{ij} \geq 0 & \perp & U_{ij}\lambda_i \leq p'_j + 1, & \forall i \in A, j \in G \\ p'_j \geq 0 & \perp & \sum_{i\in A} f_{ij} \leq p'_j + 1, & \forall j \in G \\ \lambda_i \geq 0 & \perp & -\sum_{j\in G} f_{ij} - z \leq -\sum_j W_{ij}(p'_j + 1), & \forall i \in A \\ z \geq 0 & & & \end{array} . \quad (20)$$

Note that we are interested in finding a solution of (20) with $z = 0$. Let $y = (\lambda, p', f)$ and if we set $y = 0$ in (20) then we get an infinite edge which has one endpoint $(y = 0, z = \infty)$ and other where $(y = 0, z = \max_i \sum_j W_{ij})$. We now apply the Lemke's algorithm starting with the primary ray and Complementarity pivoting (see Section 4.1 for Lemke's algorithm). It can be shown that there are no secondary rays in (20), which will imply that Lemke's algorithm will converge to a market equilibrium. For the proof, we refer to [11, 13].

5.3 Convex Programs

In last section, we obtained an LCP formulation and a complementary pivot algorithm (using Lemke's algorithm) for computing a market equilibrium. Even though this algorithm runs very fast in practice, the worst case complexity may be exponential (like in the case of Simplex algorithm for the linear programming). In this section, we discuss convex programming based techniques which are used to design provably efficient algorithms for computing equilibria in linear markets.

For linear Fisher markets, the following Eisenberg-Gale (EG) convex program [12] captures equilibria:

$$\begin{aligned} \max \sum_{i\in A} M_i \log \sum_{j\in G} U_{ij}x_{ij} \\ \sum_{i\in A} x_{ij} \leq 1, \ \forall j \in G, \\ x_{ij} \geq 0, \ \forall i \in A, j \in G \end{aligned}$$

where prices p_j's are dual variables of the first constraint. The Karush-Kuhn-Tucker (KKT) conditions that are satisfied at an optimal solution of the convex program are as follows, where p_j and α_{ij} are dual variables of the two constraints respectively:

$$\begin{aligned} \frac{M_i U_{ij}}{\sum_k U_{ik}x_{ik}} = p_j + \alpha_{ij}, \quad & \forall i, j \\ x_{ij}\alpha_{ij} = 0, \quad & \forall i, j \\ p_j(\sum_i x_{ij} - 1) = 0, \quad & \forall j \end{aligned}$$

These together with the primal constraints, we get that the optimal solution of EG convex program satisfies the following:

$$\begin{array}{lll} \frac{M_i U_{ij}}{\sum_k U_{ik} x_{ik}} \le p_j & \perp & x_{ij} \ge 0, \quad \forall i,j \\ \sum_i x_{ij} \le 1 & \perp & p_j \ge 0, \quad \forall j \end{array}$$

Rearranging the first constraint we get

$$\begin{array}{lll} \frac{U_{ij}}{p_j} \le \frac{\sum_k U_{ik} x_{ik}}{M_i} & \perp & x_{ij} \ge 0, \quad \forall i,j \\ \sum_i x_{ij} \le 1 & \perp & p_j \ge 0, \quad \forall j \end{array}$$

Note that the right hand side of the first constraint is independent of good j, and hence it implies that whenever $x_{ij} > 0$ we have $U_{ij}/p_j = \max_l U_{il}/p_l$. This implies that at an optimal solution of the EG convex program, each agent receives an optimal bundle. Next, the first constraint also implies that (using complementarity and multiplying by x_{ij} on both sides)

$$U_{ij} x_{ij} M_i = x_{ij} p_j \sum_k U_{ik} x_{ik}, \ \forall i,j.$$

If we sum the above for all j, then we get

$$M_i = \sum_j x_{ij} p_j \ ,$$

which implies that at an optimal solution of the EG convex program, each agent gets an optimal bundle that is within its budget. Finally, $\sum_i x_{ij} \le 1$ constraint implies that no good is oversold. These together implies that optimal solution of the EG convex program gives a Fisher market equilibrium and vice-versa.

There is another convex program given by Shmyrev [17] which also captures Fisher market equilibria at its optimal solution (which is in fact a dual of the EG convex program):

$$\begin{array}{ll} \min & \sum_{j \in G} p_j \log p_j - \sum_{ij:U_{ij}>0} f_{ij} \log U_{ij} \\ & \sum_{i:U_{ij}>0} f_{ij} = p_j \quad \forall j \in G \\ & \sum_{j:U_{ij}>0} f_{ij} = B_i \quad \forall i \in A \\ & f_{ij} \ge 0, p_j \ge 0 \quad \forall i \in A, j \in G \end{array}$$

The Shmyrev convex program can be interpreted as a min cost network flow problem as shown in Figure 1.

An equilibrium in a linear Fisher market can be computed in $O(n^4 \log n)$ time where n is the total number of buyers and goods [16, 18].

For the linear exchange markets, it can be assumed without loss of generality that each agent i brings a unit amount of good i. For that, [9] gave the following

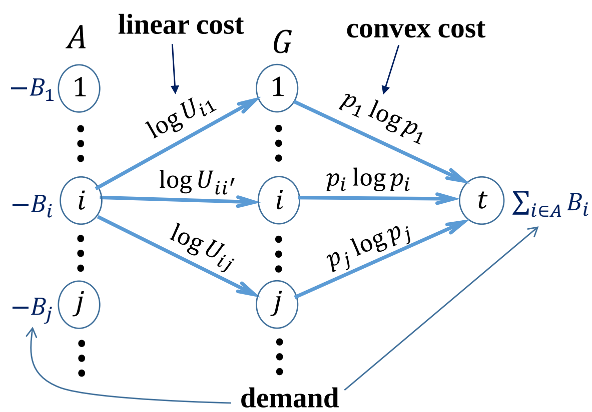

Figure 1: Network flow interpretation of Shmyrev's convex program

convex program:

$$\min \sum_{i\in A} p_i \log \frac{p_i}{\beta_i} - \sum_{ij:U_{ij}>0} f_{ij} \log U_{ij}$$
$$\sum_{i:U_{ij}>0} f_{ij} = p_j, \quad \forall j \in G$$
$$\sum_{j:U_{ij}>0} f_{ij} = p_i, \quad \forall i \in A$$
$$U_{ij}\beta_i \le p_j, \quad \forall i \in A, j \in G$$
$$f_{ij} \ge 0, \beta_i \ge 0, p_j \ge 1, \quad \forall i \in A, j \in G$$

An equilibrium in a linear exchange market can be computed in polynomial time, and the fastest algorithm is given by [10].

5.4 Welfare Theorems

Given a linear exchange market $\mathcal{M}$, let $p = (p_j)_{j\in G}, x = (x_1, \ldots, x_n)$ be an equilibrium, where p_j is the price of good j and x_i is the allocation to agent i. Let X denote the set of feasible allocation, i.e.,

$$X = \{(x_1, \ldots, x_n) \mid \sum_i x_{ij} \le 1\}.$$

Clearly, $x \in X$. Consider another feasible allocation $x' = (x'_1, \ldots, x'_n)$. Then, agent i either prefers x_i to x'_i or it prefers x'_i to x_i. In the former case, equilibrium allocation is better and in the latter case equilibrium allocation is worse. Observe that in the latter case $x'_i.p > x_i.p$ (where $x_i.p = \sum_j x_{ij} p_j$) because market equilibrium gives an optimal bundle to each agent subject to budget constraints and hence the bundle x'_i was not affordable (or within the budget of agent i).

Definition 5.6 (Pareto Efficiency)**.** *A feasible allocation $x \in X$ is said to be Pareto efficient (or just efficient) if there is no feasible allocation $x' \in X$ such that $x'_i \geq x_i, \forall i$ and it is a strict inequality for at least one agent i.*

Theorem 5.7 (First Welfare Theorem)**.** *Market equilibrium allocations are efficient.*

Proof. By contradiction. Let (p, x) be a market equilibrium and there exists x' such that each agent i is either indifferent between x_i and x'_i or prefers x'_i to x_i, and there is at least one agent i who prefers x'_i to x_i. In that case, we have

$$\sum_j p_j = \sum_i x'_i.p > \sum_i x_i.p = \sum_j p_j,$$

which is a contradiction. □

Theorem 5.8 (Second Welfare Theorem)**.** *Every Pareto efficient allocation can be obtained as a market equilibrium.*

Proof. Let x be an efficient allocation (in a market with respect to the preferences of the agents). If we consider each agent i endowment W_i to be same as x_i, then we claim that x is an equilibrium allocation in this market. Suppose not and let x' is an equilibrium of such a market. In that case, since $x \neq x'$ and x is efficient, there exists some agent i who prefers x_i to x'_i. But since x_i is the endowment of agent i, it should get a bundle that is at least as better as x_i, which is a contradiction. □

5.5 Applications

5.5.1 Fair Division

In fair division, the problem is divide a given a set of items (or goods) into a set of agents in a *fair* way. One notion of fairness is an allocation that is Pareto efficient and *envy-free*.

Definition 5.9 (Envy-free)**.** *An allocation $x = (x_1, \ldots, x_n)$ is said to be envy-free if no agent i prefers x_j to x_i for some $j \neq i$.*

Note that this is not a market problem. But, suppose all goods are divisible (i.e., they can be fractionally divided, e.g., milk, cake, etc.) then if we convert this problem into a market problem where each agent has \$1 (equal income) and their utility function is same as the preference function in the fair division problem. Then we claim that such an allocation is envy-free and efficient and hence fair.

Lemma 5.10 (Competitive equilibrium with equal income)**.** *Market equilibrium gives a fair allocation.*

Proof. We need to show that market equilibrium allocation $x = (x_1, \ldots, x_n)$ is efficient and envy-free. The efficiency is shown in Theorem 5.7. For the envy-freeness, since each agent i has \$1 each and they get an optimal (i.e., utility maximizing) bundle at an equilibrium. Hence, $x_j, \forall j$ is within budget for each agent i and hence x_j is not preferred to x_i by each agent i, otherwise i is not obtaining an optimal bundle, which is a contradiction. □

The above lemma shows that in case of divisible goods, fair division problem is exactly equal to market equilibrium where each agent has equal income.

In case of indivisible items, the fair division problem turns out to be infeasible. Consider an example where there are two agents and two goods, one good is preferable to other good by both agents, say a diamond and a petty stone. Clearly, there is no way these two goods can be divided between two agents so that the allocation is envy-free, i.e., whoever gets the stone will envy other.

We need to relax this notion to make some sense. Budish [5] showed that if we consider *envy-free upto the removal of one item* (or in short EF1), i.e., can there exist an allocation in any fair division problem where each agent i prefers (or indifferent) its own bundle x_i to the agent j's bundle minus one item (i.e., after removing one item from x_j) for each j. If this is true for each i, then we say that this allocation is EF1.

Nash social welfare. Another notion of fairness in case of indivisible items is Nash social welfare (NSW) which allocates so that the geometric mean of valuations is maximized, i.e.,

$$\max_{x \in X} (\prod_i v_i(x_i))^{1/n},$$

where X is the set of all feasible allocation and v_i is the valuation function of agent i. Let U_{ij} is the value of item j to agent i. In case of additive valuations (or linear), $v_i(x_i) = \sum_j U_{ij} x_{ij}$. So the NSW problem is the following integral convex program:

$$\begin{array}{lr} \max(\prod_i \sum_j U_{ij} x_{ij})^{1/n} & \\ \sum_i x_{ij} \le 1, & \forall j \\ x_{ij} \in \{0, 1\}, & \forall i, j \end{array}, \tag{21}$$

which is same as

$$\begin{array}{lr} \max \sum_i \log \sum_j U_{ij} x_{ij} & \\ \sum_i x_{ij} \le 1, & \forall j \\ x_{ij} \in \{0, 1\}, & \forall i, j \end{array}, \tag{22}$$

If we relax the integrality constraint in the above program then observe that it is same as the Eisenberg-Gale convex program where $M_i = 1, \forall i$. The NSW problem is NP-hard, and one way to design a good approximation algorithm (say obtain an allocation efficiently (in polynomial time) which is at least one half of the optimum) is to relax the integrality constraint and then round the fractional solution. This approach directly doesn't work but with some additional modification in the Fisher market equilibrium problem, it gives a 2-approximation algorithm; see [7, 6, 3] for details).

5.5.2 Proportional Response Dynamics

In P2P network (for file sharing etc., e.g., in BitTorrent) each node shares its upload bandwidth in the proportion of the bandwidth it receives from its neighbors. Let w_i denote the upload bandwidth of node i, $\Gamma(i)$ denote the neighbors of node i, and $x_{ij}(t)$ denote the fraction of upload bandwidth of node j allocated to node i at time t. Then, the allocation at time $t+1$ is:

$$x_{ij}(t+1) = \frac{x_{ji}(t)w_i}{\sum_{k\in\Gamma(j)} x_{jk}(t)w_k}.$$

This dynamics is called proportional response, and the question here is whether this dynamics converges and if yes, then to what allocation it converges to.

Wu and Zhang [20] showed that it converges to market equilibrium of the following linear exchange market, where each node i is an agent i and it brings w_i amount of good i. Its utility from a bundle x_i is $\sum_{j\in\Gamma(i)} w_j x_{ij}$. This is remarkable because we can easily find the steady allocation of the proportional response dynamics from equilibrium of the linear exchange market.

5.6 Tatonnement

Tatonnement is a dynamics given by Leon Walras in 1874 [19] in order to explain how market converges to an equilibrium. There is an auctioneer who has all the goods. It initializes the prices of all goods to some value (say all 1). And at these prices, it obtains the total demand of each good. If demand is equal to supply of each good, then trade happens at these prices (which are in fact equilibrium prices). Otherwise, trade doesn't happen and the prices of those goods whose demand is more than supply are increased and the prices of those goods whose demand is less than supply are decreased. Then, the process repeats.

It is clear that such a process (or dynamics) will converge to an equilibrium. However, the question is whether this will eventually converge or not in every market. It turns out that this may not converge in all markets, but if the market satisfies the *gross substitute* condition, then it converges in those markets.

Definition 5.11 (Gross Substitute). *A market is said to satisfy the gross substitute condition if when prices are increased for a subset $S \subset G$ of goods, then the demand of goods whose prices are not increased (i.e., in $G \setminus S$ increases.*

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